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Faces of Relativistic Toda Chain

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ABSTRACT

We demonstrate that the generalization of the relativistic Toda chain (RTC) is a special reduction of two-dimensional Toda Lattice hierarchy (2DTL). This reduction implies that the RTC is gauge equivalent to the discrete AKNS hierarchy and, which is the same, to the two-component Volterra hierarchy while its forced (semi-infinite) variant is described by the unitary matrix integral. The integrable properties of the RTC hierarchy are revealed in different frameworks of: Lax representation, orthogonal polynomial systems, and τ-function approach. Relativistic Toda molecule hierarchy is also considered, along with the forced RTC. Some applications to biorthogonal polynomial systems are discussed.

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1 Introduction

Since the paper of Ruijsenaars [1], where has been proposed, the relativistic Toda chain (RTC) system was investigated in many papers [2, 3, 4, 5]. This system can be defined by the equation:

$$\begin{aligned} \ddot{q}_n = & \ (1+\epsilon\dot{q}_n)(1+\epsilon\dot{q}_{n+1})\frac{\exp(q_{n+1}-q_n)}{1+\epsilon^2\exp(q_{n+1}-q_n)} \ - \\ - & \ (1+\epsilon\dot{q}_{n-1})(1+\epsilon\dot{q}_n)\frac{\exp(q_n-q_{n-1})}{1+\epsilon^2\exp(q_n-q_{n-1})} \end{aligned} \tag{1.1}$$

which transforms to the ordinary (non-relativistic) Toda chain (TC) in the evident limit $\epsilon \to 0$. The RTC is integrable, which was discussed in different frameworks (see, for example, [2, 3, 4, 5] and references therein). In particular, in [2] there was introduced a Lax triad for the RTC, in the paper [3] there were investigated Lax representation and its orbit interpretation for the RTC and the relativistic Toda molecule, and also suggested a discrete version of the RTC which has the same integrals of motion (i.e. the same Lax operator, but different evolution). At last, authors of [5] presented the bilinear (Hirota type) equations whose solutions satisfy the RTC, implying the possibility of τ-function interpretation of the model. Thus, the RTC can be obtained from many different starting points. In particular, this is a limit of the general Ruijsenaars system [1].

On the other hand, it seems that the proper place of the RTC in the framework of the general theory of the integrable systems has not been adequately understood yet. Since the seminal papers of Kyoto school [6] it is evident that any integrable system can be treated as a particular reduction of the (multi-component) Kadomtsev-Petviashvili hierarchy. This statement is certainly true for the RTC too. Using the powerful machinery of [6], it is possible to describe wide classes of solutions to the RTC and reveal many interesting algebraic properties of the system. Another important lesson of the same philosophy teaches us that many would-be different systems have the common structures behind if considered on the level of fundamental notions like Baker-Akhiezer functions, Grassmannian manifolds etc. In this paper we want to advocate these points more (in somehow reversed order). Indeed, we shall show that the RTC has the deep connection with many well known integrable systems. Namely, the special case of the AKNS hierarchy [7] as well as the discrete non-linear Schrödinger equation (of type 2) [7, 8] are nothing but different faces of the RTC. Some "novel hierarchy" considered recently [9] is equivalent to RTC also; we shall see that all these systems belong to the same class of equivalence: the corresponding L-operators are connected with each other by the suitable gauge transformations. Another interesting observation is that the forced (i.e. semi-infinite) variant of the RTC is described by the unitary matrix model which, in turn, naturally arises in the context of the two-dimensional gravity [10, 11]. Actually, the later case was our starting point. Namely, we have started from the integrable system (with the infinite set of the integrable flows) [12] inspired by the unitary matrix model. The equivalence with the discrete AKNS hierarchy was established there. Here we show that this system is nothing but the generalized version of the RTC hierarchy.

The connection with matrix models [1] turns out to be important by some reasons. First of all, it allows one to apply effectively the orthogonal polynomials technique, which considerably simplifies all the derivations and proofs. Indeed, in the systems of such a type there are two different (but essentially equivalent) Lax representations - those given by 2×2 matrices depending on the spectral parameter and given by infinite matrices (in spirit of [13]). We mostly base on the second representation, which is effectively treated by orthogonal polynomials methods (since they solve the corresponding Riemann-Hilbert problem – see [11]). As a by-product of our consideration, we obtain some new results specifically interesting for the orthogonal polynomials theory. Second, it was shown in [12] that unitary matrix model is equivalent to the two-component Volterra system, thus establishing the connection between this latter and the RTC. On the other hand, Volterra systems are well investigated and, therefore, this last connection might help to get more about the RTC system from this different point of view.

There is another even more useful representation of the RTC. Namely, we demonstrate in this paper that the RTC is a reduction of the two-dimensional (subholonomic[2]) Toda lattice hierarchy (2DTL). On this level, the equivalence of the above mentioned systems is especially transparent. Looking at the RTC as a reduction of the 2DTL also allows us to give some τ-function interpretation of the RTC and realize it in terms of fermionic correlators. This finally implies the bilinear identities like those proposed in [5] and the theory group interpretation. Let us also stress that the τ-function (Grassmannian) interpretation being absolutely algebraic can be easily deformed to the quantum case (along the line of the papers [14],[15]) [3]. Furthermore, using group-theoretical approach one can deal with different RTC's - that is, the general RTC, forced RTC and RTC molecule hierarchy - from the algebraic point of view. Their algebraic properties are very interesting. In particular, the latter hierarchy is related to a class of co-representations of $U_q(SL(N))$ in the quasi-classical limit. Put it differently, in the Hamiltonian approach

[1] Let us stress that the correspondence of the RTC and a unitary matrix model is not so surprising: it is well-known that the Hermitean matrix model describes the TC hierarchy [11]. On the other hand, the RTC is usually considered as a group generalization of the TC whatever it means. Therefore, it is natural to identify the RTC with some unitary matrix model.

[2] See [13].

[3] Quantization of the RTC was considered in [4], where the Sklyanin's scheme of the separation of variables was applied.

the relativistic Toda molecule is given as the integrable system on some Poisson leaves on $SL(N)$ [16] (see [17] for mathematical background). This is why we consider this approach as a very promising one.

This paper is organized as follows. In section 2 we discuss the simple spectral problem (three-term recurrent relation) for the RTC which is natural generalization of the corresponding problem for the usual Toda Chain (TC). Then, we introduce two *different* flows which lead to the *same* equation (1.1) and interpret them in the context of the results by Suris [3]. These results, we guess, are not new and can be partly extracted from preceeding papers (see, for example, [3, 4]). The main aim of section 2 is, however, to establish the key points which allow us to connect all the integrable structures mentioned above.

Section 3 contains the *derivation* of the both RTC spectral problem (proposed in Section 2) and the evolution equations from the (bi)orthogonality conditions implied by the unitary matrix model. As a by-product of the orthogonality conditions, remarkably, we automatically get the spectral problem in such a form that describes the connection of the Baker-Akhiezer functions of the two-dimensional Toda Lattice hierarchy. We show that the solution to the RTC hierarchy satisfies simultaneously the first 2DTL equation with respect two the flows introduced in the previous section. The connection with the two-component Volterra hierarchy as well as with the AKNS system is also discussed. As a consequence, we get the direct correspondence between the RTC, a particular discrete version of the non-linear Schrödinger equation, and the "novel" integrable system [9]. At the end of the section, we reveal the existence of *two* Lax operators for the RTC which naturally arise from an immediate generalization of the recurrent relations inspired by the unitary matrix model.

In section 4 we outline the essential ingredients of the 2DTL theory [13] and explicitly describe the whole RTC hierarchy as a particular reduction of 2DTL.

The forced RTC hierarchy and its finite analog, the relativistic Toda molecule, are discussed in sections 5 and 6 respectively. We give the group interpretation of the later hierarchy. We also discuss the fermionic representations for the τ-functions of the corresponding hierarchies and manifestly indicate the reductions from the 2DTL which describe the hierarchies in terms of the Grassmannian. We present also the explicit solution to the forced RTC in the determinant form. The fermionic representation received allows one to make some connection with the results of [5], but we postpone the discussion of this point till a separate publication [19] (see, however, section 9).

In section 7 we describe the simple approach to discrete evolutions of the RTC which is based on the notion of the Darboux-Bäcklund transformations and can be considered as a natural generalization of the corresponding notion in the usual Toda chain theory. The continuum limit as well as the limit to the Toda chain is also discussed here. We consider also some degeneration of the general Darboux-Bäcklund transformation leading to simplified discrete equations which can be treated as a kind of "modified Toda chain" (these equations have been proposed for the first time in [3] but we should stress that they have nothing to do with the usual discrete-time Toda chain since the Lax operators of these two systems are different).

Section 8 is devoted to some applications to the theory of biorthogonal polynomials. Indeed, we discuss the (relativistic) orthogonal polynomials which satisfy the three-term recurrent relation specific for the RTC and look at some very specific solutions to these relations. One of these solutions describes finite sets of the polynomials corresponding to the Toda molecule, the other one is a special limit of the Askey polynomials [18]. In the section, we discuss the orthogonality measures and some additional orthogonality relations which can be manifestly treated in the indicated simple cases.

In concluding remarks we briefly discuss more general (and, in a sense, more natural) description of the RTC hierarchy which will be considered in detail in the forthcoming publication [19].

In Appendices we give the proof of the some important assertions used in the main body of the paper. In particular, in Appendix A there were obtained the evolution equations starting from the orthogonality conditions determining the unitary matrix model, while in Appendix B the spectral problem naturally arising in the framework of orthogonal polynomials is transformed to that of the 2DTL hierarchy.

2 Lax representation for RTC

In this section we describe the Lax representation for the standard RTC equation. The usual procedure to obtain integrable non-linear equations consists of the two essential steps:

i) To find appropriate spectral problem for the Baker-Akhiezer function(s).

ii) To define the proper evolution of this function with respect to isospectral deformations.

2.1 Lax representation by three-term recurrent relation

In the theory of the usual Toda chain the first step implies the discretized version of the Schrödinger equation (see [8], for example). In order to get the relativistic extension of the Toda equations, one should consider the following "unusual" spectral problem

$$\Phi_{n+1}(z) + a_n\Phi_n(z) = z\{\Phi_n(z) + b_n\Phi_{n-1}(z)\} \quad , \qquad n \in \mathbb{Z} \tag{2.1}$$

representing a particular discrete Lax operator acting on the Baker-Akhiezer function $\Phi_n(z)$. This is a simple three-term recurrent relation (similar to those for the Toda and Volterra chains) but with "unusual" spectral dependence.

As for the second step, one should note that there exist *two* distinct integrable flows leading to the same equation (1.1). As we shall see below, the spectral problem (2.1) can be naturally incorporated into the theory of two-dimensional Toda lattice (2DTL) which describes the evolution with respect to two (infinite) sets of times $(t_1, t_2, ...)$, $(t_{-1}, t_{-2}, ...)$ (positive and negative times, in accordance with [13]). In section 4, we shall derive the general evolution equations satisfied by Φ_n. Here we describe the two particular flows (at the moment, we deal with them "by hands", i.e. introducing the corresponding Lax pairs by a guess) which lead to the RTC equations (1.1). It turns out (as usual) that the simplest equations are to be associated with the first times of this hierarchy t_1 and t_{-1}. The most simple evolution equation is that with respect to the first *negative* time and has the form

$$\frac{\partial\Phi_n}{\partial t_{-1}} = R_n\Phi_{n-1} \tag{2.2}$$

with some (yet unknown) R_n . Remarkably enough that the form of this equation exactly coincides with the corresponding equation for the non-relativistic Toda chain.

The compatibility condition determines R_n in terms of a_n and b_n:

$$R_n = \frac{b_n}{a_n} \tag{2.3}$$

and leads to the following equations of motion:

$$\frac{\partial a_n}{\partial t_{-1}} = \frac{b_n}{a_{n-1}} - \frac{b_{n+1}}{a_{n+1}} \tag{2.4}$$

$$\frac{\partial b_n}{\partial t_{-1}} = b_n\left(\frac{1}{a_{n-1}} - \frac{1}{a_n}\right) \tag{2.5}$$

In order to get (1.1), we should identify

$$a_n = \exp(-\epsilon p_n) \tag{2.6}$$

$$b_n = -\epsilon^2\exp(q_n - q_{n-1} - \epsilon p_n) \tag{2.7}$$

Note that in this parameterization the "Hamiltonian" R_n in (2.3) depends only on coordinates q_n's:

$$R_n = -\epsilon^2\exp(q_n - q_{n-1}) \tag{2.8}$$

Such a form is typical for integrable systems (for example, the first flow of the TC has the same structure).

In order to preserve the proper limit to the TC equations as $\epsilon \to 0$, one should perform the rescaling of time in (2.4), (2.5)

$$t_{-1} \to \nu(\epsilon)t_{-1} \tag{2.9}$$

where the function $\nu(\epsilon)$ is to behave as $1/\epsilon$ in the limit of small ϵ. Then, from (2.4), (2.5)[4]

$$\frac{\partial q_n}{\partial t_{-1}} = -\nu(\epsilon)\Big\{1 + \epsilon^2\ \exp(q_{n+1} - q_n)\Big\}\exp(\epsilon p_n) + \nu(\epsilon)(1+\epsilon^2) \tag{2.10}$$

$$\frac{\partial p_n}{\partial t_{-1}} = \epsilon\nu(\epsilon)\Big\{\exp(q_n - q_{n-1} + \epsilon p_{n-1}) - \exp(q_{n+1} - q_n + \epsilon p_n)\Big\} \tag{2.11}$$

After choosing

$$\nu = -[\epsilon(1+\epsilon^2)]^{-1} \tag{2.12}$$

[4]Strictly speaking, (2.5) gives the equation $\frac{\partial q_n}{\partial t_{-1}} - \frac{\partial q_{n-1}}{\partial t_{-1}} = Q_n - Q_{n-1}$, where Q_n is the exponential term in (2.10). Thus, one can only write

$$\frac{\partial q_n}{\partial t_{-1}} = -\nu(\epsilon)\{1 + \epsilon^2\ \exp(q_{n+1} - q_n)\}\exp(\epsilon p_n) + constant$$

For the fast-decreasing solutions, *constant* $= \nu(\epsilon)(1+\epsilon^2)$. This choice of the constant nicely agrees with the non-relativistic limit $\epsilon \to 0$: in this limit $\frac{\partial q_n}{\partial t_{-1}} = p_n$.

these equations trivially lead to the RTC equation

$$\begin{aligned}\frac{\partial^2 q_n}{\partial t_{-1}^2} &= \left(1+\epsilon\frac{\partial q_{n+1}}{\partial t_{-1}}\right)\left(1+\epsilon\frac{\partial q_n}{\partial t_{-1}}\right)\frac{\exp(q_{n+1}-q_n)}{1+\epsilon^2\exp(q_{n+1}-q_n)} \quad - \\ &- \left(1+\epsilon\frac{\partial q_n}{\partial t_{-1}}\right)\left(1+\epsilon\frac{\partial q_{n-1}}{\partial t_{-1}}\right)\frac{\exp(q_n-q_{n-1})}{1+\epsilon^2\exp(q_n-q_{n-1})}\end{aligned} \tag{2.13}$$

Another form of the same equation is given by the replace

$$\begin{aligned}\xi_n &= \frac{1}{a_n} = \exp(\epsilon p_n) \\ \eta_n &= \frac{b_n}{a_n a_{n-1}} = -\epsilon^2\exp(q_n - q_{n-1} + \epsilon p_{n-1})\end{aligned} \tag{2.14}$$

Then, from (2.4), (2.5) [5]

$$\begin{aligned}\frac{\partial \xi_n}{\partial t_{-1}} &= \xi_n(\eta_{n+1}-\eta_n) \\ \frac{\partial \eta_n}{\partial t_{-1}} &= \eta_n(\eta_{n+1}-\eta_{n-1}+\xi_{n-1}-\xi_n)\end{aligned} \tag{2.15}$$

As we noted already, the evolution (2.2), which leads to the RTC equations is not the unique one. The other possible choice leading to the same equations is

$$\frac{\partial \Phi_n}{\partial t_1} = -b_n(\Phi_n - z\Phi_{n-1}) \tag{2.16}$$

The compatibility condition of (2.1) and (2.16) gives the equations

$$\frac{\partial a_n}{\partial t_1} = -a_n(b_{n+1}-b_n) \tag{2.17}$$

$$\frac{\partial b_n}{\partial t_1} = -b_n(b_{n+1}-b_{n-1}+a_{n-1}-a_n) \tag{2.18}$$

which are exactly of the form (2.15). In terms of $(q_n,\ p_n)$ we have the following equations (with the same rescaling of time as in (2.9)):

$$\frac{\partial q_n}{\partial t_1} = -\nu(\epsilon)\Big\{1+\epsilon^2\ \exp(q_n-q_{n-1})\Big\}\exp(-\epsilon p_n) + \nu(\epsilon)(1+\epsilon^2) \tag{2.19}$$

$$\frac{\partial p_n}{\partial t_1} = \epsilon\nu(\epsilon)\Big\{\exp(q_{n+1}-q_n-\epsilon p_{n+1}) - \exp(q_n-q_{n-1}-\epsilon p_n)\Big\} \tag{2.20}$$

slightly different from (2.10), (2.11). Nevertheless, the second order equation for q_n is exactly (2.13).

2.2 2×2 matrix Lax representation

The same RTC equation can be obtained from the matrix Lax operator depending on the spectral parameter [3] (generalizing the Lax operator for the TC [8]). Then the RTC arises as the compatibility condition for the following 2×2 matrix equations:

$$L_n^{(S)}\psi_n = \psi_{n+1} \quad , \quad \frac{\partial \psi_n}{\partial t} = A_n\psi_n \tag{2.21}$$

where

$$L_n^{(S)} = \begin{pmatrix} \zeta\exp(\epsilon p_n)-\zeta^{-1} & \epsilon\exp(q_n) \\ -\epsilon\exp(-q_n+\epsilon p_n) & 0\end{pmatrix} \quad ; \quad \psi_n = \begin{pmatrix}\psi_n^{(1)} \\ \psi_n^{(2)}\end{pmatrix} \tag{2.22}$$

$$A_n = \begin{pmatrix} \epsilon^2\exp(q_n-q_{n-1}+\epsilon p_{n-1}) & -\epsilon\zeta^{-1}\exp(q_n) \\ \epsilon\zeta^{-1}\exp(-q_{n-1}+\epsilon p_{n-1}) & 1-\ \zeta^{-2}+\epsilon^2\end{pmatrix} \tag{2.23}$$

Using (2.22), one can re-write the matrix spectral problem as the recurrent relation for $\psi_n^{(1)}$:

$$\psi_{n+1}^{(1)}(\zeta) = \Big\{\zeta\ \exp(\epsilon p_n) - \zeta^{-1}\Big\}\psi_n^{(1)}(\zeta) - \epsilon^2\exp\Big(q_n-q_{n-1}+\epsilon p_{n-1}\Big)\psi_{n-1}^{(1)}(\zeta) \tag{2.24}$$

Then, putting

$$\psi_n^{(1)}(\zeta) \equiv \mathcal{N}_n\Phi_n(\zeta) \tag{2.25}$$

[5] In the paper [4] $d_n = \xi_n$, $c_n = -\eta_{n+1}$, $t = -t_{-1}$ and $\epsilon = 1$.

where

$$\frac{\mathcal{N}_n}{\mathcal{N}_{n+1}} = \zeta \exp(-\epsilon p_n) \tag{2.26}$$

one gets

$$\Phi_{n+1}(\zeta) + \exp(-\epsilon p_n)\Phi_n(\zeta) = \zeta^2 \Big\{ \Phi_n(\zeta) - \epsilon^2 \exp\Big(q_n - q_{n-1} - \epsilon p_n\Big)\Phi_{n-1}(\zeta) \Big\} \tag{2.27}$$

This is exactly recurrent relation (2.1) with a_n, b_n given by (2.6), (2.7) after the identification

$$z = \zeta^2 \tag{2.28}$$

In the same manner, the evolution equation in terms of Φ_n takes the form

$$\frac{\partial \Phi_n}{\partial t} = \epsilon^2 \exp(q_n - q_{n-1})\Phi_{n-1} \tag{2.29}$$

Thus, the evolution determined by (2.23) is associated with $-t_{-1}$ -flow in our approach (see (2.2), (2.8)).

Obviously, we are able to construct another A-operator which generates a new integrable flow:

$$A'_n = \begin{pmatrix} 0 & -\epsilon\zeta \exp(q_n - \epsilon p_n) \\ \epsilon\zeta \exp(-q_{n-1}) & \zeta^2 + \epsilon^2 \exp(q_n - q_{n-1} - \epsilon p_n) - 1 - \epsilon^2 \end{pmatrix} \tag{2.30}$$

It is easy to see that (2.30) gives the evolution equivalent to (2.16) on the level of the Baker-Akhiezer function Φ_n.

To conclude this section, we should remark that L-operator (2.22), which determines the RTC is not unique; moreover, it is not the simplest one. Indeed, in the next section we shall see that there exists the whole family of the gauge equivalent operators, which contains more "natural" ones and includes, in particular, the well known operator generating the AKNS hierarchy. From general point of view, the whole RTC hierarchy is nothing but AKNS and vice versa.

3 RTC and unitary matrix model, AKNS, etc.

Now we are going to describe the generalized RTC hierarchy as well as its connection with some other integrable systems. We start our investigation from the framework of orthogonal polynomials. The advantage of it is that one does not need to *guess* Lax pair, but instead can *get* it automatically using the formalism developed in [11, 12, 20]. From the point of view of the integrable systems, this method is nothing but the Riemann-Hilbert problem [11, 20]. The orthogonal polynomials give the solutions only for hierarchies of the forced type (see the proper definition below). Fortunately, the hierarchies we consider are "local" ones (one-dimensional subfamilies of the subholonomic systems [13]) either can be reduced to these, and, therefore, Lax representations obtained from the orthogonal polynomials formalism can be continued to the whole hierarchies [20]. From now on, we mostly arbitrarily turn from the forced hierarchies to the general ones and back. However, right in the RTC case, this point still deserves some comments which we postpone till section 5.

3.1 Unitary matrix model

It is well-known that the partition function τ_n of the unitary one-matrix model can be presented as a product of norms of the biorthogonal polynomial system [10, 21]. Namely, let us introduce a scalar product of the form [6]

$$< A, B >= \oint \frac{d\mu(z)}{2\pi i z} \exp\Big\{ \sum_{m>0} (t_m z^m - t_{-m} z^{-m}) \Big\} A(z) B(\frac{1}{z}) \tag{3.1}$$

where the integration measure is not fixed but to be chosen so that the integral in (3.1) is well defined; for example, if $d\mu(z)$ can be presented in the form $dz \exp(\sum c_k z^k)$ then everything reduces to the usual measure dz with the proper shifts of times t_k , t_{-k}. Let us define the system of polynomials biorthogonal with respect to this scalar product

$$< \Phi_n, \Phi^\star_k >= h_n \delta_{nk} \tag{3.2}$$

Then, the partition function τ_n of the unitary matrix model is equal to the product of h_n's:

$$\tau_n = \prod_{k=0}^{n-1} h_k \ , \qquad \tau_0 \equiv 1 \tag{3.3}$$

[6]The signs of positive and negative times are defined in this way to get the exact correspondence with the times introduced in [13].

the integration contour in (3.1) being the unit circle [7] (and, therefore, h_n's are real for the real times $t_k,\ t_{-k}$'s). The polynomials are normalized as follows (we should stress that superscript '$\star$' does not mean the complex conjugation):

$$\Phi_n(z) = z^n + \ldots + S_{n-1}, \qquad \Phi_n^\star(z) = z^n + \ldots + S_{n-1}^\star, \qquad S_{-1} = S_{-1}^\star \equiv 1 \tag{3.4}$$

Now it is easy to show that these polynomials satisfy the following recurrent relations, which gives the discrete Lax operator:

$$\begin{aligned} \Phi_{n+1}(z) &= z\Phi_n(z) + S_n z^n \Phi_n^\star(z^{-1}) \\ \Phi_{n+1}^\star(z^{-1}) &= z^{-1}\Phi_n^\star(z^{-1}) + S_n^\star z^{-n}\Phi_n(z) \end{aligned} \tag{3.5}$$

and

$$\frac{h_{n+1}}{h_n} = 1 - S_n S_n^\star \tag{3.6}$$

These recurrent relations can be written in several equivalent forms. First, it can be presented in the form analogous to (2.1), i.e. equivalent to the spectral problem for the RTC:

$$\Phi_{n+1} - \frac{S_n}{S_{n-1}}\Phi_n = z\left\{\Phi_n - \frac{S_n}{S_{n-1}}(1 - S_{n-1}S_{n-1}^\star)\Phi_{n-1}\right\} \tag{3.7}$$

$$\Phi_{n+1}^\star - \frac{S_n^\star}{S_{n-1}^\star}\Phi_n^\star = z^{-1}\left\{\Phi_n^\star - \frac{S_n^\star}{S_{n-1}^\star}(1 - S_{n-1}S_{n-1}^\star)\Phi_{n-1}^\star\right\} \tag{3.8}$$

From the first relation and using (2.27), one can immediately read off

$$\begin{aligned} \frac{S_n}{S_{n-1}} &= -\exp(-\epsilon p_n) \\ \frac{h_n}{h_{n-1}} &= -\epsilon^2 \exp(q_n - q_{n-1}) \end{aligned} \tag{3.9}$$

i.e.

$$h_n = \gamma(-\epsilon^2)^n \exp\Big(q_n\Big) \tag{3.10}$$

where γ is a constant which does not depend on n. Thus, the orthogonality conditions (3.2) lead exactly to the spectral problem for the RTC. We should stress that equations (3.7), (3.8) can be *derived* from system (3.5). Usually, the three-term recurrent relations can be transformed to the system of two equations in many different ways (see below). The main feature of (3.5), which distinguishes it from the other possible choices is the remarkable fact: this system describes the connection between the pair of the Baker-Akhiezer functions arising in the context of 2DTL. We shall discuss this point in section 4.

Some remarks are in order now. In all formulas above the discrete index n runs over the non-negative integers. One can trivially extend all relations (3.5)-(3.10) (and all the relations below) to arbitrary $n \in \mathbb{Z}$. Then, the polynomial case corresponds to the conditions

$$\frac{h_n}{h_{n-1}} \equiv 0 \ , \quad n < 0 \tag{3.11}$$

or, (see (3.9))

$$q_{-1} = \infty \tag{3.12}$$

with (formal) ordering $q_{-1} < q_{-2} < \ \ldots$. Equivalently, the analytical continuation of (3.3) written in the form

$$h_n = \frac{\tau_{n+1}}{\tau_n} \tag{3.13}$$

reduces to the polynomial case by imposing the conditions

$$\tau_n \equiv 0 \ , \quad n\ < 0 \tag{3.14}$$

This is exactly what we call the *forced* integrable systems [20]. Our way of doing is to derive all the formulas in the forced ($\equiv$ polynomial) case and then to extend them to arbitrary values of the discrete indices.

Using the orthogonal conditions, it is also possible to obtain the equations which describe the time dependence of $\Phi_n,\ \Phi_n^\star$. Namely, differentiating (3.2) with respect to times t_1 , t_{-1} gives the following evolution equations (for derivation see Appendix A):

$$\frac{\partial\Phi_n}{\partial t_1} = \frac{S_n}{S_{n-1}}\frac{h_n}{h_{n-1}}(\Phi_n - z\Phi_{n-1}) \tag{3.15}$$

$$\frac{\partial\Phi_n}{\partial t_{-1}} = \frac{h_n}{h_{n-1}}\Phi_{n-1} \tag{3.16}$$

$$\frac{\partial\Phi_n^\star}{\partial t_1} = -\frac{h_n}{h_{n-1}}\Phi_{n-1}^\star \tag{3.17}$$

[7] We do not restrict ourselves to this integration contour throughout this paper.

$$\frac{\partial \Phi_n^\star}{\partial t_{-1}} = -\frac{S_n^\star}{S_{n-1}^\star}\frac{h_n}{h_{n-1}}(\Phi_n^\star - z^{-1}\Phi_{n-1}^\star) \tag{3.18}$$

(see general evolution equations with respect to higher flows in the next section). The compatibility conditions give the following nonlinear evolution equations [8]:

$$\frac{\partial S_n}{\partial t_1} = S_{n+1}\frac{h_{n+1}}{h_n} \tag{3.19}$$

$$\frac{\partial S_n}{\partial t_{-1}} = S_{n-1}\frac{h_{n+1}}{h_n} \tag{3.20}$$

$$\frac{\partial S_n^\star}{\partial t_1} = -S_{n-1}^\star\frac{h_{n+1}}{h_n} \tag{3.21}$$

$$\frac{\partial S_n^\star}{\partial t_{-1}} = -S_{n+1}^\star\frac{h_{n+1}}{h_n} \tag{3.22}$$

As a consequence, in the polynomial case,

$$\frac{\partial h_n}{\partial t_1} = -S_n S_{n-1}^\star h_n \tag{3.23}$$

$$\frac{\partial h_n}{\partial t_{-1}} = S_{n-1}S_n^\star h_n \tag{3.24}$$

These are exactly relativistic Toda equations written in somewhat different form. Indeed, from (3.23), (3.19) and (3.21) one gets

$$\frac{\partial^2}{\partial t_1^2}\log h_n = -S_{n+1}S_{n-1}^\star\frac{h_{n+1}}{h_n} + S_n S_{n-2}^\star\frac{h_n}{h_{n-1}} \tag{3.25}$$

Using (3.23) again and (3.6)

$$\left(\frac{\partial}{\partial t_1}\log h_n\right)\left(\frac{\partial}{\partial t_1}\log h_{n-1}\right) = S_n S_{n-2}^\star\left(1-\frac{h_n}{h_{n-1}}\right) \tag{3.26}$$

Substitution of $S_n S_{n-2}^\star$ to (3.25) gives [9]

$$\begin{aligned}\frac{\partial^2}{\partial t_1^2}\log h_n = &-\left(\frac{\partial}{\partial t_1}\log h_n\right)\left(\frac{\partial}{\partial t_1}\log h_{n+1}\right)\frac{\frac{h_{n+1}}{h_n}}{1-\frac{h_{n+1}}{h_n}} + \\ &+\left(\frac{\partial}{\partial t_1}\log h_{n-1}\right)\left(\frac{\partial}{\partial t_1}\log h_n\right)\frac{\frac{h_n}{h_{n-1}}}{1-\frac{h_n}{h_{n-1}}}\end{aligned} \tag{3.27}$$

On the other hand, the RTC is a particular case of the 2DTL hierarchy. Indeed, let us introduce the key objects in the theory of integrable systems - the τ-function as it is defined in (3.13). Then, with the help of (3.19)-(3.24), one can show that the functions τ_n satisfy the first equation of the 2DTL:

$$\partial_{t_1}\partial_{t_{-1}}\log\tau_n = -\frac{\tau_{n+1}\tau_{n-1}}{\tau_n^2} \tag{3.28}$$

Therefore, it is natural to assume that the higher flows generate the whole set of non-linear equations of the 2DTL in spirit of [13]. In the next section, we shall see that this is indeed the case. On the other hand, it turns out that the integrable system determined by (3.7), (3.8) is highly degenerate as compared with the general 2DTL. Indeed, the solutions to the equations (3.7), (3.8) (treated for a moment as two separate equations) correspond to the four Baker-Akhiezer functions of the 2DTL. In the general theory [13], these functions are linearly independent. In the RTC case, they are dependent due to (3.5); this is the origin of the degeneracy. Actually, this degeneracy is responsible for the appearance of additional non-linear equations, which are absent in typical situation. For example, these are equations (3.27), which contain the derivatives of $\log h_n$ with respect to t_1 (or t_{-1}) only. These additional equations being consistent with the whole 2DTL hierarchy specialize the reduction of the 2DTL. Therefore, we should treat the RTC hierarchy as a special reduction of an "abstract" 2DTL (see section 4). [10]

Let us also note that, using (3.10), one can get from equation (3.27) the equation distinguished from (1.1) by the linear (in time) shift of the variable $q_n \to q_n - \frac{1}{\epsilon}t_1$. This means that one should careful when express q_n through τ-functions.

This completes the derivation of the RTC from the unitary matrix model. Now we consider the connection of the RTC with the integrable systems mentioned in the Introduction.

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[8] In the polynomial case, one can get these equations directly from (3.15)-(3.18) considering terms of the order $\sim z^0$; see definition (3.4).

[9] The same equation holds for t_{-1}-flow.

[10] Let us stress that the term "reduction" does not obligatory mean the polynomial case since non-polynomial solutions to (3.7), (3.8), (3.15)- (3.18) are also reduced in the above mentioned sense.

3.2 Two-component Volterra

Recursion relations (3.7)–(3.8) can be presented in the form establishing their connection with the two-component Volterra hierarchy [12] (it should be understood that, generally, Φ_n and $\Phi_n^\star$ in these relations are not polynomials). For doing this, we introduce the two-component *functions*

$$f_n^{(\pm)}(z) \equiv z^{-\frac{n}{2}+1}\Phi_{n-1}(z) \pm z^{\frac{n}{2}-1}\Phi_{n-1}^\star(z^{-1}) \tag{3.29}$$

These functions satisfy the following orthogonality conditions:

$$\begin{aligned} < f_n^{(\pm)}, f_k^{(\pm)} > &= \left[2 \mp (S_{n-1} + S_{n-1}^\star)\right] h_{n-1}\delta_{nk} \\ < f_n^{(+)}, f_k^{(-)} > &= (S_{n-1} - S_{n-1}^\star) h_{n-1}\delta_{nk} \end{aligned} \tag{3.30}$$

These functions also satisfy the recurrent relations which can be written in the compact matrix form after introducing the two-component column

$$f_n \equiv \begin{pmatrix} f_n^{(+)} \\ f_n^{(-)} \end{pmatrix} \tag{3.31}$$

Then, one gets the recurrent relations giving the two-component Volterra Lax operator

$$\begin{aligned} (z^{\frac{1}{2}} + z^{-\frac{1}{2}}) f_n &= f_{n+1} + V_n f_{n-1} \\ (z^{\frac{1}{2}} - z^{-\frac{1}{2}}) f_n &= \sigma_1 f_{n+1} + \tilde{V}_n f_{n-1} \end{aligned} \tag{3.32}$$

where

$$V_n \equiv W_n W_{n-1}^{-1} \quad , \quad \tilde{V}_n \equiv W_n \sigma_3 W_{n-1}^{-1} \tag{3.33}$$

σ_i's are Pauli matrices and

$$W_n = h_{n-1} \begin{pmatrix} 1 - S_{n-1} & 1 - S_{n-1}^\star \\ -(1 + S_{n-1}) & 1 + S_{n-1}^\star \end{pmatrix} \quad ; \quad \det W_n = 2 h_n h_{n-1} \tag{3.34}$$

Three forms of the recurrent relations (3.5), (3.7) and (3.32) explicitly establish the connection between the unitary matrix model, RTC and two-component Volterra system (certainly, there still remains to compare the evolutions, see the next section).

3.3 RTC versus AKNS and "novel" hierarchies

Now let us demonstrate the correspondence between RTC and AKNS system. We have already seen that the orthogonality conditions naturally lead to the 2×2 formulation of the problem generated by the unitary matrix model:

$$L^{(\mathrm{U})} \begin{pmatrix} \Phi_n \\ \Phi_n^\star \end{pmatrix} = \begin{pmatrix} \Phi_{n+1} \\ \Phi_{n+1}^\star \end{pmatrix} \quad , \quad L^{(\mathrm{U})} = \begin{pmatrix} z & z^n S_n \\ z^{-n} S_n^\star & z^{-1} \end{pmatrix} \tag{3.35}$$

$$\frac{\partial}{dt_1} \begin{pmatrix} \Phi_n \\ \Phi_n^\star \end{pmatrix} = \begin{pmatrix} -S_n S_{n-1}^\star & z^n S_n \\ z^{1-n} S_{n-1}^\star & -z \end{pmatrix} \begin{pmatrix} \Phi_n \\ \Phi_n^\star \end{pmatrix} \tag{3.36}$$

$$\frac{\partial}{dt_{-1}} \begin{pmatrix} \Phi_n \\ \Phi_n^\star \end{pmatrix} = \begin{pmatrix} z^{-1} & -z^{n-1} S_{n-1} \\ -z^{-n} S_n^\star & S_{n-1} S_n^\star \end{pmatrix} \begin{pmatrix} \Phi_n \\ \Phi_n^\star \end{pmatrix} \tag{3.37}$$

(Equations (3.36), (3.37) follow from (3.15)-(3.18) and the original spectral problem (3.5)). Put

$$\begin{aligned} \Phi_n &\equiv z^{n/2-1/4} F_n \\ \Phi_n^\star &\equiv z^{-n/2+1/4} F_n^\star \end{aligned} \tag{3.38}$$

Then the spectral problem (3.35) can be rewritten in the matrix form

$$L_n^{(\mathrm{AKNS})} \mathcal{F}_n = \mathcal{F}_{n+1} \quad , \quad \mathcal{F}_n \equiv \begin{pmatrix} F_n \\ F_n^\star \end{pmatrix} \tag{3.39}$$

where

$$L_n^{(\mathrm{AKNS})} = \begin{pmatrix} \zeta & S_n \\ S_n^\star & \zeta^{-1} \end{pmatrix} \quad , \quad \zeta \equiv z^{1/2} \tag{3.40}$$

This is the Lax operator for the discrete AKNS [7]. Obviously the evolution equations (3.36), (3.37) can be written in terms of F_n, $F_n^\star$ as

$$\frac{\partial \mathcal{F}_n}{\partial t_1} = A_n^{(1)} \mathcal{F}_n \ , \qquad A_n^{(1)} = \begin{pmatrix} -S_n S_{n-1}^\star & \zeta S_n \\ \zeta S_{n-1}^\star & -\zeta^2 \end{pmatrix} \tag{3.41}$$

$$\frac{\partial \mathcal{F}_n}{\partial t_{-1}} = - A_n^{(-1)} \mathcal{F}_n \ , \qquad A_n^{(-1)} = \begin{pmatrix} -\zeta^{-2} & \zeta^{-1} S_{n-1} \\ \zeta^{-1} S_n^\star & -S_{n-1} S_n^\star \end{pmatrix} \tag{3.42}$$

Note that after introducing the trivial flow

$$\frac{\partial \mathcal{F}_n}{\partial t_0} = A_n^{(0)} \mathcal{F}_n \ , \qquad A_n^{(0)} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{3.43}$$

we get the difference non-linear Schrödinger system (DNLS) [7] (see also [8]) generated by the "mixed" flow

$$\begin{aligned} \frac{\partial \mathcal{F}_n}{\partial T} &\equiv \Big(\frac{\partial}{\partial t_0} - \frac{\partial}{\partial t_{-1}} - \frac{\partial}{\partial t_1} \Big) \mathcal{F}_n = (A_n^{(0)} + A_n^{(-1)} - A_n^{(1)}) \mathcal{F}_n \equiv \\ &\equiv \begin{pmatrix} 1 + S_n S_{n-1}^\star - \zeta^{-2} & \zeta^{-1} S_{n-1} - \zeta S_n \\ \zeta^{-1} S_n^\star - \zeta S_{n-1}^\star & -1 - S_{n-1} S_n^\star + \zeta^2 \end{pmatrix} \end{aligned} \tag{3.44}$$

Indeed, from the compatibility conditions for (3.40), (3.44) or, equivalently, just from (3.19)-(3.20) (along with the trivial evolution $\partial_{t_0} S_n = 2S_n$, $\partial_{t_0} S_n^\star = -2S_n^\star$) one gets the discrete version of the nonlinear Schrödinger equation:

$$\frac{\partial S_n}{\partial T} = -(S_{n+1} - 2S_n + S_{n-1}) + S_n S_n^\star \big(S_{n+1} + S_{n-1} \big) \tag{3.45}$$

Note also that the "novel" hierarchy of [9] is equivalent to the RTC (and, therefore, to the AKNS hierarchy) as well. Namely, the Lax operator in [9], i.e.

$$\widehat{L}_n = \begin{pmatrix} z + u_n v_n & u_n \\ v_n & 1 \end{pmatrix} \quad ; \qquad \widehat{L}_n \begin{pmatrix} \phi_n^{(1)} \\ \phi_n^{(2)} \end{pmatrix} = \begin{pmatrix} \phi_{n+1}^{(1)} \\ \phi_{n+1}^{(2)} \end{pmatrix} \tag{3.46}$$

defines the recurrent relation of the form (3.7):

$$\phi_{n+1}^{(1)} - \Big(u_n v_n + \frac{u_n}{u_{n-1}} \Big) \phi_n^{(1)} = z \left(\phi_n^{(1)} - \frac{u_n}{u_{n-1}} \phi_{n-1}^{(1)} \right) \tag{3.47}$$

thus revealing the connection with the RTC. Comparing (3.7) and (3.47) leads to the identification

$$u_n = S_n h_n \ , \qquad v_n = \frac{S_{n-1}^\star}{h_n} \tag{3.48}$$

where h_n's satisfy (3.6). Moreover, from (3.46) and (3.5) it is easy to see that

$$\begin{aligned} \phi_n^{(1)} &= \Phi_n \\ \phi_n^{(2)} &= \frac{1}{h_n} \Big(z^n \Phi_n^\star - S_{n-1}^\star \Phi_n \Big) \end{aligned} \tag{3.49}$$

and, therefore, $\widehat{L}_n$ can be obtained from $L_n^{(\mathrm{AKNS})}$ by the discrete gauge transformation:

$$\widehat{L}_n = U_{n+1} L_n^{(\mathrm{AKNS})} U_n^{-1} \tag{3.50}$$

(for $z = \zeta^2$) where

$$U_n = z^{n/2 - 1/4} \begin{pmatrix} 1 & 0 \\ -\frac{S_{n-1}^\star}{h_n} & \frac{z^{1/2}}{h_n} \end{pmatrix} \tag{3.51}$$

Evolution equations (3.19)- (3.24) in terms of new variables (3.48) have the form

$$\begin{aligned} \frac{\partial u_n}{\partial t_1} &= u_{n+1} - u_n^2 v_n \ , \qquad & \frac{\partial u_n}{\partial t_{-1}} &= \frac{u_{n-1}}{1 + u_{n-1} v_n} \\ \frac{\partial v_n}{\partial t_1} &= -v_{n-1} + u_n v_n^2 \ , \qquad & \frac{\partial v_n}{\partial t_{-1}} &= -\frac{v_{n+1}}{1 + u_n v_{n+1}} \end{aligned} \tag{3.52}$$

and easily reproduce the usual AKNS equations in the continuum limit since

$$\begin{aligned} \Big(\partial_{t_0} - \partial_{t_1} - \partial_{t_{-1}} \Big) u_n &= -\Big(u_{n+1} - 2u_n + u_{n-1} \Big) + \Big(u_{n-1}^2 + u_n^2 \Big) v_n + \ldots \\ \Big(\partial_{t_0} - \partial_{t_1} - \partial_{t_{-1}} \Big) v_n &= \Big(v_{n+1} - 2v_n + v_{n-1} \Big) - \Big(v_n^2 + v_{n+1}^2 \Big) u_n + \ldots \end{aligned} \tag{3.53}$$

We conclude with the remark that the operator $L_n^{(\mathrm{S})}$ in (2.22) is also gauge equivalent to $L_n^{(\mathrm{AKNS})}$:

$$L_n^{(\mathrm{S})} = \widetilde{U}_{n+1} L_n^{(\mathrm{AKNS})} \widetilde{U}_n^{-1} \tag{3.54}$$

where

$$\widetilde{U}_n = \begin{pmatrix} \dfrac{(-1)^n}{S_{n-1}} & 0 \\ \dfrac{\epsilon^{2n-1} z^{-1/2}}{S_{n-1} h_n} & -\dfrac{\epsilon^{2n-1}}{h_n} \end{pmatrix} \tag{3.55}$$

3.4 Non-local Lax representation

At the end of the section let us note that there is another form of the recurrent relations which is non-local (i.e. contains all the functions with smaller indices) but instead expresses $\Phi_n(z)$ through themselves. This form of the spectral problem will be crucial for dealing with the RTC as a particular reduction of the 2DTL. Let us introduce the normalized functions

$$\mathcal{P}_n(z) \equiv \Phi_n(z) \quad , \qquad \mathcal{P}_n^\star(z^{-1}) \equiv \frac{1}{h_n} \Phi_n^\star(z^{-1}) \tag{3.56}$$

such that, for example, in the polynomial case

$$\langle \mathcal{P}_n, \mathcal{P}_k^\star \rangle = \delta_{nk} \tag{3.57}$$

Then, from (3.7), (3.8) one can show that in the forced and fast-decreasing cases some proper solutions (see the discussion in Appendix B) satisfy the equations

$$\begin{gathered} z\mathcal{P}_n(z) = \mathcal{P}_{n+1}(z) - S_n h_n \sum_{k=-\infty}^{n} \frac{S_{k-1}^\star}{h_k} \mathcal{P}_k(z) \equiv \mathcal{L}_{nk} \mathcal{P}_k(z) \\ z^{-1} \mathcal{P}_n^\star(z^{-1}) = \frac{h_{n+1}}{h_n} \mathcal{P}_{n+1}^\star(z^{-1}) - S_n^\star \sum_{k=-\infty}^{n} S_{k-1} \mathcal{P}_k^\star(z^{-1}) \equiv \overline{\mathcal{L}}_{kn} \mathcal{P}_k^\star(z^{-1}) \end{gathered} \tag{3.58}$$

Note that this expression is correct for general (non-polynomial) $\mathcal{P}_n$ and $\mathcal{P}_n^\star$ provided the sums run over all integer k. In the polynomial case, the sums automatically run over only non-negative k. Let us also stress the natural appearance of variables (3.48) in the Lax operator $\mathcal{L}_{nk}$. The last representation of the spectral problem will be useful in the next section to determine the general evolution of the system. Indeed, these relations manifestly describe the embedding of the RTC into the 2DTL [13, 12], which is given essentially by *two* Lax operators ($\mathcal{L}$ and $\overline{\mathcal{L}}$).

4 RTC as reduction of 2DTL

4.1 Hierarchy evolution

In order to determine the whole set of the evolution equations, one can use different tricks. The simplest one is to use the orthogonal polynomial technique and then to continue the result for the "general" hierarchy. We use this way at the end of the section. Now let us note that there is also some less direct way to obtain the evolution suitable even in the non-polynomial case without continuation. Namely, one can use embedding (3.58) of the system into the 2DTL, making use of the standard evolution of this latter [13]. Let us briefly describe the formalism of the 2DTL following [13]. In their framework, one introduces *two* different Baker-Akhiezer (BA) $\mathbb{Z} \times \mathbb{Z}$ matrices $\mathcal{W}$ and $\overline{\mathcal{W}}$. (In the original paper [13], these matrices were denoted as $W^{(\infty)}$ and $W^{(0)}$; the superscripts correspond to the marked points on the Riemann surface which specify the solution to the 2DTL, i.e. the point of the infinite dimensional Grassmannian). These matrices satisfy the linear system:

i) the matrix version of the spectral problem:

$$\mathcal{L}\mathcal{W} = \mathcal{W}\Lambda \quad , \quad \overline{\mathcal{L}}\, \overline{\mathcal{W}} = \overline{\mathcal{W}} \Lambda^{-1} \tag{4.1}$$

ii) the matrix version of the evolution equations:

$$\begin{aligned} &\frac{\partial \mathcal{W}}{\partial t_m} = (\mathcal{L}^m)_+ \mathcal{W} \, , \qquad \frac{\partial \overline{\mathcal{W}}}{\partial t_m} = (\mathcal{L}^m)_+ \overline{\mathcal{W}} \\ &\frac{\partial \mathcal{W}}{\partial t_{-m}} = (\overline{\mathcal{L}}^{\,m})_- \mathcal{W} \, , \qquad \frac{\partial \overline{\mathcal{W}}}{\partial t_{-m}} = (\overline{\mathcal{L}}^{\,m})_- \overline{\mathcal{W}} \, , \qquad m = 1, 2, \ldots \end{aligned} \tag{4.2}$$

where $\mathbb{Z}\times\mathbb{Z}$ matrices $\mathcal{L}$ and $\overline{\mathcal{L}}$ have (by definition) the following structure:

$$\mathcal{L}=\sum_{i\le 1}\mathrm{diag}[b_i(s)]\Lambda^i \;\;;\quad b_1(s)=1$$
$$\overline{\mathcal{L}}=\sum_{i\ge -1}\mathrm{diag}[c_i(s)]\Lambda^i \;\;;\quad c_{-1}(s)\neq 0 \tag{4.3}$$

Here $\mathrm{diag}[b_i(s)]$ denotes an infinite diagonal matrix $\mathrm{diag}(\ldots\ b_i(-1),\ b_i(0),\ b_i(1),\ \ldots)$; Λ is the shift matrix with the elements $\Lambda_{nk}\equiv\delta_{n,k-1}$ and for arbitrary infinite matrix $A=\sum_{i\in\mathbb{Z}}\mathrm{diag}[a_i(s)]\Lambda^i$ we set

$$(A)_+\equiv\sum_{i\ge 0}\mathrm{diag}[a_i(s)]\Lambda^i \;\;,\quad (A)_-\equiv\sum_{i<0}\mathrm{diag}[a_i(s)]\Lambda^i \tag{4.4}$$

i.e. $(A)_+$ is the upper triangular part of the matrix A (including the main diagonal) while $(A)_-$ is strictly the lower triangular part.

Note that (4.3) can be written in components as

$$\mathcal{L}_{nk}=\delta_{n+1,k}+b_{k-n}(n)\theta(n-k)\;,\quad \overline{\mathcal{L}}_{nk}=c_{-1}(n)\delta_{n-1,k}+c_{k-n}(n)\theta(k-n)\;\;;\quad n,k\in\mathbb{Z} \tag{4.5}$$

The compatibility conditions imposed on (4.1),(4.2) give rise to the infinite set (hierarchy) of nonlinear equations for the operators $\mathcal{L}$, $\overline{\mathcal{L}}$ or, equivalently, for the coefficients $b_m(n)$, $c_m(n)$. This is what is called 2DTL hierarchy.

On the level of nonlinear equations, one does not need the information of the structure of BA matrices. However, in order to get the touch with the polynomials, this information is essential. It was proved in [13] that the linear system (4.1), (4.2) is resolved by the following BA matrices:

$$\mathcal{W}=\mathbf{W}\exp\left\{\sum_{m=1}^{\infty}t_m\Lambda^m\right\}\quad,\quad \overline{\mathcal{W}}=\overline{\mathbf{W}}\exp\left\{\sum_{m=1}^{\infty}t_{-m}\Lambda^{-m}\right\} \tag{4.6}$$

where $\mathbf{W}$, $\overline{\mathbf{W}}$ can be presented in the form:

$$\mathbf{W}\equiv\sum_{i=0}^{\infty}\mathrm{diag}[w_i(s)]\Lambda^{-i}\;,\quad \mathbf{W}^{-1}\equiv\sum_{i=0}^{\infty}\Lambda^{-i}\,\mathrm{diag}[w_i^\star(s+1)]\;,\quad w_0(s)=w_0^\star(s)\equiv 1 \tag{4.7}$$

$$\overline{\mathbf{W}}\equiv\sum_{i=0}^{\infty}\mathrm{diag}[\overline{w}_i(s)]\Lambda^{i}\;,\quad \overline{\mathbf{W}}^{\,-1}\equiv\sum_{i=0}^{\infty}\Lambda^{i}\mathrm{diag}[\overline{w}_i^{\,\star}(s+1)] \tag{4.8}$$

Now let us introduce the BA *functions* as follows:

$$w_n(z)\equiv\sum_{k\in\mathbb{Z}}\mathbf{W}_{nk}z^k=z^n\sum_{i=0}^{\infty}w_i(n)z^{-i} \tag{4.9}$$

$$\overline{w}_n(z)\equiv\sum_{k\in\mathbb{Z}}\overline{\mathbf{W}}_{nk}z^k=z^n\sum_{i=0}^{\infty}\overline{w}_i(n)z^{i} \tag{4.10}$$

At the same time, we define the *adjoint* BA functions as

$$zw_{n+1}^\star(z)\equiv\sum_{k\in\mathbb{Z}}\mathbf{W}^{-1}_{kn}z^{-k}\;\;;\quad w_n^\star(z)=z^{-n}\sum_{i=0}^{\infty}w_i^\star(n)z^{-i} \tag{4.11}$$

$$z\overline{w}_{n+1}^{\,\star}(z)\equiv\sum_{k\in\mathbb{Z}}\overline{\mathbf{W}}^{\,-1}_{kn}z^{-k}\;\;;\quad \overline{w}_n^{\,\star}(z)=z^{-n}\sum_{i=0}^{\infty}\overline{w}_i^{\,\star}(n)z^{i} \tag{4.12}$$

Using (4.1), (4.2), it is easy to show that the BA functions satisfy the linear equations:

$$\mathcal{L}_{nk}w_k(z)=zw_n(z)\;\;,\quad \overline{\mathcal{L}}_{nk}\overline{w}_k(z)=z^{-1}\overline{w}_n(z) \tag{4.13}$$

$$\frac{\partial w_n(z)}{\partial t_m}=-[(\mathcal{L}^m)_-]_{nk}w_k(z)\;,\qquad \frac{\partial\overline{w}_n(z)}{\partial t_m}=[(\mathcal{L}^m)_+]_{nk}\overline{w}_k(z)$$
$$\frac{\partial w_n(z)}{\partial t_{-m}}=[(\overline{\mathcal{L}}^{\,m})_-]_{nk}w_k(z)\;,\qquad \frac{\partial\overline{w}_n}{\partial t_{-m}}=-[(\overline{\mathcal{L}}^{\,m})_+]_{nk}\overline{w}_k(z)\;,\qquad m=1,2,\ldots \tag{4.14}$$

Corresponding system for the adjoint BA functions is:

$$\mathcal{L}_{kn}w_{k+1}^\star(z)=zw_{n+1}^\star(z)\;\;,\quad \overline{\mathcal{L}}_{kn}\overline{w}_{k+1}^{\,\star}(z)=z^{-1}\overline{w}_{n+1}^{\,\star}(z) \tag{4.15}$$

$$\frac{\partial w_{n+1}^\star(z)}{\partial t_m}=[(\mathcal{L}^m)_-]_{kn}w_{k+1}^\star(z)\;,\qquad \frac{\partial\overline{w}_{n+1}^{\,\star}(z)}{\partial t_m}=-[(\mathcal{L}^m)_+]_{kn}\overline{w}_{k+1}^{\,\star}(z)$$
$$\frac{\partial w_{n+1}^\star(z)}{\partial t_{-m}}=-[(\overline{\mathcal{L}}^{\,m})_-]_{nk}w_{k+1}^\star(z)\;,\qquad \frac{\partial\overline{w}_{n+1}^{\,\star}}{\partial t_{-m}}=[(\overline{\mathcal{L}}^{\,m})_+]_{kn}\overline{w}_{k+1}^{\,\star}(z)\;,\qquad m=1,2,\ldots \tag{4.16}$$

4.2 Reduction from 2DTL

Now let us return to the case of RTC "in general situation", i.e. with the conditions $S_n = 1, S_n^\star = 1 \;\; n < 0$ restricting onto the polynomial case being removed away. From (3.58), one gets two matrices

$$\mathcal{L}_{nk} = \delta_{n+1,k} - \frac{h_n}{h_k} S_n S_{k-1}^\star \theta(n-k) \;\; , \qquad k, n \in \mathbb{Z} \tag{4.17}$$

$$\overline{\mathcal{L}}_{nk} = \frac{h_n}{h_{n-1}} \delta_{n-1,k} - S_{n-1} S_k^\star \theta(k-n) \;\; , \qquad k, n \in \mathbb{Z} \tag{4.18}$$

which have exactly the form (4.5). One can consider the spectral problem (4.13) for these particular operators. It is easy to prove that every solution to (4.13), (4.17), (4.18) satisfies the recurrent equation (3.7). On the other hand, (3.7) has two linear independent solutions with asymptotics (we are using the identification (3.56))

$$\mathcal{P}_n^{(1)}(z) = z^n \left(1 + O\left(\frac{1}{z}\right) \right) \qquad z \to \infty \tag{4.19}$$

$$\mathcal{P}_n^{(2)}(z) = h_n z^n \left(1 + O(z) \right) \qquad z \to 0 \tag{4.20}$$

which are precisely the same as asymptotics of (4.9) and (4.10) respectively. Thus, the solutions to (3.7) should be identified with the corresponding BA functions $w_n(z)$ and $\overline{w}_n(z)$:

$$w_n(z) \equiv \mathcal{P}_n^{(1)}(z) \tag{4.21}$$

$$\overline{w}_n(z) \equiv \mathcal{P}_n^{(2)}(z) \tag{4.22}$$

Similarly, one can consider the spectral problem (4.15), (4.17), (4.18). Then, the corresponding $w_n^\star(z)$ and $\overline{w}_n^\star(z)$ satisfy the same three-term recurrent equation as $\mathcal{P}_n^\star(z^{-1}) \equiv \Phi_n^\star(z^{-1})/h_n$ (see (3.56) and (3.8)), i.e.

$$\frac{h_{n+1}}{h_n} \mathcal{P}_{n+1}^\star - \frac{S_n^\star}{S_{n-1}^\star} \mathcal{P}_n^\star = z^{-1} \left\{ \mathcal{P}_n^\star - \frac{S_n^\star}{S_{n-1}^\star} \mathcal{P}_{n-1}^\star \right\} \tag{4.23}$$

This equation obviously has two independent solutions with asymptotics

$$\mathcal{P}_n^{\star(1)}(z^{-1}) = z^{-n} \left(1 + O\left(\frac{1}{z}\right) \right) \qquad z \to \infty \tag{4.24}$$

$$\mathcal{P}_n^{\star(2)}(z^{-1}) = \frac{1}{h_n} z^{-n} \left(1 + O(z) \right) \qquad z \to 0 \tag{4.25}$$

which are exactly of the form (4.11) and (4.12). Thus, the adjoint BA functions should be identified with solutions to (4.23) as follows:

$$w_n^\star(z) = z^{-1} \mathcal{P}_{n-1}^{\star(1)}(z^{-1}) \tag{4.26}$$

$$\overline{w}_n^\star(z) = z^{-1} \mathcal{P}_{n-1}^{\star(2)}(z^{-1}) \tag{4.27}$$

To conclude, we see that equation (3.58) (see also (4.28) below) for $\mathcal{P}_n(z)$, $\mathcal{P}_n^\star(z^{-1})$ is exactly the same as that for $w_{n+1}(z)$, $\overline{w}_{n+1}^\star(z)$ (compare with (4.14), (4.16)).

4.3 Evolution and orthogonal polynomials

Let us now demonstrate how one can obtain evolution of the RTC using the technique of the orthogonal polynomials. Indeed, differentiating the orthogonality conditions with respect to arbitrary times, one can obtain with the help of (3.58) the evolution of polynomials $\mathcal{P}_n$ and $\mathcal{P}_n^\star$ (for details see Appendix A):

$$\begin{aligned} \frac{\partial \mathcal{P}_n}{\partial t_m} &= -[(\mathcal{L}^m)_-]_{nk} \mathcal{P}_k \\ \frac{\partial \mathcal{P}_n}{\partial t_{-m}} &= [(\overline{\mathcal{L}}^{\,m})_-]_{nk} \mathcal{P}_k \\ \frac{\partial \mathcal{P}_n^\star}{\partial t_m} &= -[(\mathcal{L}^m)_+]_{kn} \mathcal{P}_k^\star \\ \frac{\partial \mathcal{P}_n^\star}{\partial t_{-m}} &= [(\overline{\mathcal{L}}^{\,m})_+]_{kn} \mathcal{P}_k^\star \\ \frac{\partial h_n}{\partial t_m} = (\mathcal{L}^m)_{nn} h_n \quad &, \qquad \frac{\partial h_n}{\partial t_{-m}} = -(\overline{\mathcal{L}}^{\,m})_{nn} h_n \end{aligned} \tag{4.28}$$

where by definition for given matrix C_{nk},

$$[(C)_+]_{nk} \equiv C_{nk}\theta(k-n) \quad , \quad (C_-)_{nk} \equiv C_{nk}\theta(n-k-1) \tag{4.29}$$

i.e. $(C)_+$ is the upper triangular part of C while $(C)_-$ is strictly the lower triangular part of C .

Let us note that this evolution (of the BA functions) has the same form as (4.14), (4.16). However, the latter one was determined on the infinite matrices, while the evolution (4.28) – on the quarter part of these matrices (this is the forced hierarchy, we will return to this point later). In order to get the hierarchy of non-linear equations, one should use the compatibility conditions of the linear systems. The hierarchies obtained in this way from the two considered evolutions do not coincide exactly although differing only by inessential constants. Now we consider in more details the evolution with respect to the first times to observe this phenomenon.

Using the general formulas (4.14), (4.16) one gets the simplest evolution equations

$$\begin{aligned} \frac{\partial \Phi_n}{\partial t_1} &= S_n h_n \sum_{k=-\infty}^{n-1} \frac{S^\star_{k-1}}{h_k}\Phi_k \equiv \\ &\equiv \frac{S_n}{S_{n-1}}\frac{h_n}{h_{n-1}}(\Phi_n - z\Phi_{n-1}) \end{aligned} \tag{4.30}$$

$$\frac{\partial \Phi_n}{\partial t_{-1}} = \frac{h_n}{h_{n-1}}\Phi_{n-1} \equiv (1 - S_{n-1}S^\star_{n-1})\Phi_{n-1} \tag{4.31}$$

$$\frac{\partial \Phi^\star_n}{\partial t_1} = -\frac{h_n}{h_{n-1}}\Phi^\star_{n-1} \equiv -(1 - S_{n-1}S^\star_{n-1})\Phi^\star_{n-1} \tag{4.32}$$

$$\begin{aligned} \frac{\partial \Phi^\star_n}{\partial t_{-1}} &= -S^\star_n h_n \sum_{k=-\infty}^{n-1} \frac{S_{k-1}}{h_k}\Phi^\star_k \equiv \\ &\equiv -\frac{S^\star_n}{S^\star_{n-1}}\frac{h_n}{h_{n-1}}(\Phi^\star_n - z^{-1}\Phi^\star_{n-1}) \end{aligned} \tag{4.33}$$

The compatibility conditions give the following nonlinear equations:

$$\frac{\partial S_n}{\partial t_1} = S_{n+1}\frac{h_{n+1}}{h_n} + \alpha S_n \tag{4.34}$$

$$\frac{\partial S_n}{\partial t_{-1}} = S_{n-1}\frac{h_{n+1}}{h_n} + \beta S_n \tag{4.35}$$

$$\frac{\partial S^\star_n}{\partial t_1} = -S^\star_{n-1}\frac{h_{n+1}}{h_n} - \alpha S^\star_n \tag{4.36}$$

$$\frac{\partial S^\star_n}{\partial t_{-1}} = -S^\star_{n+1}\frac{h_{n+1}}{h_n} - \beta S^\star_n \tag{4.37}$$

$$\frac{\partial h_n}{\partial t_1} = -(S_n S^\star_{n-1} + \gamma)h_n \tag{4.38}$$

$$\frac{\partial h_n}{\partial t_{-1}} = (S_{n-1}S^\star_n + \gamma)h_n \tag{4.39}$$

From the orthogonal polynomials, one obtains the same equations with $\alpha = \beta = \gamma = 0$ [12] (see also Appendix A). This difference, however, does not look crucial by the following reason. One can use identification (3.9) and evolutions (3.19), (3.21) and (3.23) in order to get the RTC equations (2.19) and (2.20) with $\nu(\epsilon) = -1$, $\epsilon = i$ and $\gamma = 0$ (one can equally compare t_{-1}-evolutions)[11]. Let us note that the RTC itself already does not depend on α and β. Therefore, besides zero γ, one can also put α and β to be zero. Put it differently, different α, β and γ describes different representations of the same RTC hierarchy.

5 Forced RTC hierarchy

5.1 RTC-reduction of 2DTL

Now we are going to formulate in some invariant terms what reduction of the 2DTL corresponds to the RTC hierarchy. For doing this, let us return again to the Lax representation (3.58) embedding the RTC into the 2DTL. Using (3.6), one can easily

[11] The standard form of the RTC with an arbitrary value of ϵ and ν as in (2.12) can be reached by the proper redefinition of time.

prove the following identities

$$\sum_{k=n}^{N}\frac{S_{k-1}S^\star_{k-1}}{h_k}=\frac{1}{h_N}-\frac{1}{h_{N-1}}$$
$$\sum_{k=n}^{N}S_kS^\star_k h_k=h_n-h_{N+1} \tag{5.1}$$

Because of these identities, the matrices $\mathcal{L}$ and $\bar{\mathcal{L}}^T$ have zero modes $\sim S_{k-1}$ and $S^\star_{k-1}/h_k$ respectively. Therefore, one could naively expect that they are not invertible and get (using (5.1)) that

$$(\mathcal{L}\bar{\mathcal{L}})_{nk}=\delta_{nk}-\frac{S_nS^\star_k h_n}{h_{-\infty}}$$
$$(\bar{\mathcal{L}}\mathcal{L})_{nk}=\delta_{nk}-\frac{S_{n-1}S^\star_{k-1}h_\infty}{h_k} \tag{5.2}$$

Since the reduction is to be described as an invariant condition imposed on $\mathcal{L}$ and $\bar{\mathcal{L}}$, these formulas might serve as a starting point to describe the reduction of the 2DTL to the RTC hierarchy only if their r.h.s. does not depend on the dynamical variables. It seems not to be the case.

However, these formulas require some careful treatment. Indeed, the formulation of the 2DTL in terms of infinite-dimensional matrices, although being correct as a formal construction requires some accuracy if one wants to work with the genuine matrices since the products of the infinite-dimensional matrices should be properly defined. In fact, this product exists for the "band" matrices, i.e. those with only a finite number of the non-zero diagonals, and in some other more complicated cases (of special divergency conditions). One can easily see from (3.58) that the RTC Lax operators do not belong to this class. Therefore, equations (5.2) just do not make sense in this case (this is why the interpretation of the general RTC hierarchy in invariant (say, Grassmannian) terms is a little bit complicated, see the discussion in the last section).

Moreover, the presence of the zero mode does not mean automatically that the matrix is not invertible since this zero mode is to be normalizable (this is the counterpart of the band structure of matrices, the number of non-zero entries in the column describing the zero mode is to be finite, or to satisfy some fast-decreasing conditions).[12]

However, in the case of forced hierarchy, some of the indicated problems are removed since one needs to multiply only quarter-infinite matrices and, say, the product $\mathcal{L}\bar{\mathcal{L}}$ always exists. Certainly the inverse order of the multipliers is still impossible. Therefore, only the first formula in (5.2) becomes well-defined acquiring the form

$$(\mathcal{L}\bar{\mathcal{L}})_{nk}=\delta_{nk} \tag{5.3}$$

This formula can be already taken as a definition of the RTC-reduction of the 2DTL in the forced case as it does not depend on dynamical variables. Now we will show how this definition is reflected in different formulations of the 2DTL.

5.2 Fermionic representation

Here we would like to describe the 2DTL hierarchy in terms of the massless fermions [6] and to describe manifestly in the subspace in the Grassmannian corresponding to the RTC-reduction. In this and the next subsections we closely follows the papers [20] and [22] where further technical details can be found.

Let us define the fermionic operators on sphere

$$\psi(z)=\sum_{k\in\mathbf{Z}}\psi_k z^k\ ,\ \psi^*(z)=\sum_{k\in\mathbf{Z}}\psi^*_k z^{-k} \tag{5.4}$$

with fermionic modes satisfying the usual anti-commutation relations:

$$\{\psi_k,\,\psi^*_m\}=\delta_{km}\ ,\ \{\psi_k,\,\psi_m\}=\{\psi^*_k,\,\psi^*_m\}=0 \tag{5.5}$$

The Dirac vacuum $|0\rangle$ is defined by the conditions:

$$\psi_k|0\rangle=0\ ,\ k<0\ ;\ \psi^*_k|0\rangle=0\ ,\ k\geq 0 \tag{5.6}$$

We also need to introduce the "shifted" vacua

$$\psi_k|n\rangle=0\ ,\ k<n\ ;\ \psi^*_k|n\rangle=0\ ,\ k\geq n \tag{5.7}$$

[12]It is evident from the following simple example. Let us consider the band matrix $B\equiv I-\Lambda$. This matrix is invertible, but its inverse matrix $I+\Lambda+\Lambda^2+\dots$ does not possess the band structure. At the same time, there exists the zero mode $f_j=\text{const}$ of the matrix B_{ij}, but it is non-normalizable since the product $f^Tf=\left(\sum_{j=-\infty}^{+\infty}\text{const}\right)$ is divergent.

From fermionic modes one can built the $U(1)$–currents

$$J_k = \sum_{i \in \mathbf{Z}} \psi_i \psi^*_{i+k} \ , \ J_{-k} \equiv \bar{J}_k \ , \ k \in \mathbf{Z}_+ \tag{5.8}$$

and define "Hamiltonians"

$$H(t_k) = \sum_{k=1}^{\infty} t_k J_k \ , \ \bar{H}(t_{-k}) = \sum_{k=1}^{\infty} t_{-k} \bar{J}_k \tag{5.9}$$

where $\{t_k\}$ and $\{t_{-k}\}$ are "positive" and "negative" times correspondingly, which generate the evolution of nonlinear system.

Let g be an arbitrary element of the Clifford group which does not mix the ψ- and ψ^*- modes :

$$g =: \exp \left[\sum A_{km} \psi_k \psi^*_m \right] : \tag{5.10}$$

where : : denotes the normal ordering with respect to the Dirac vacuum $|0\rangle$. Then it is well known (see [22] and references therein) that

$$\tau_n(t) = \langle n | e^{H(t_k)} g e^{-\bar{H}(t_{-k})} | n \rangle \tag{5.11}$$

solves the two-dimensional Toda lattice hierarchy, i.e. is the solution to the whole set of the Hirota bilinear equations. Any particular solution depends only on the choice of the element g (or, equivalently, it can be uniquely described by the matrix A_{km}). From relations (5.5) one can conclude that any element of the form (5.10) rotates the fermionic modes as follows

$$g \psi_k g^{-1} = \psi_j R_{jk} \ , \ g \psi^*_k g^{-1} = \psi^*_j R^{-1}_{kj} \tag{5.12}$$

where the matrix R_{jk} can be expressed through A_{jk}. We will see below that the general solution (5.11) can be expressed in the determinant form with explicit dependence on R_{jk} .

Now we deal with the forced hierarchies, i.e. impose the condition

$$\tau_n = 0 \ , \ n < 0 \tag{5.13}$$

Let us determine what substitutes the general expression (5.10) in this case. It is reasonable to look for the point of the Grassmannian in the form

$$g = g_0 P_+ \tag{5.14}$$

where P_+ is the projector onto positive states:

$$P_+ |n\rangle = \theta(n) |n\rangle \tag{5.15}$$

It can be constructed from the fermionic modes as follows

$$P_+ =: \exp \left(\sum_{i<0} \psi_i \psi^*_i \right) : \tag{5.16}$$

and enjoy the properties

$$P_+ \psi^*_{-k} = \psi_{-k} P_+ = 0 \ , \ k > 0 \ ; \ [P_+, \psi_k] = [P_+, \psi^*_k] = 0 \ , \ k \geq 0 \ ; \ P_+^2 = P_+ \tag{5.17}$$

The insertion of such a projector into eq.(5.11) leaves us with g_0 depending only on ψ_k and ψ^*_k with $k \geq 0$. Therefore, it has the form [22]

$$g_0 =: \exp \left\{ \left(\int_\gamma A(z,w) \psi_+(z) \psi^*_+(w^{-1}) dz dw \right) - \sum_{i \geq 0} \psi_i \psi^*_i \right\} : \tag{5.18}$$

where $\psi_+(z) = \sum_{k \geq 0} \psi_k z^k$, $\psi^*_+(z) = \sum_{k \geq 0} \psi^*_k z^{-k}$ and γ is some integration contour. The matrix elements A_{ij} from (5.10) are immediately connected with the modes of the function $A(z,w)$ and, in the forced case, this matrix coincides with the quarter of the matrix R_{ij} from (5.12) (the rest part of R_{ij} is just unit). Now we are ready to describe manifestly the RTC reduction of the forced hierarchy. The reduction condition (5.3) means that the matrix A_{ij} celebrates the property

$$\Lambda_+ \cdot A \cdot \Lambda_- = A \tag{5.19}$$

where Λ_+ and Λ_- are the quarter-infinite matrices, $(\Lambda_+)_{ij} \equiv \delta_{i,j-1}$, $(\Lambda_-)_{ij} \equiv \delta_{i-1,j}$. These matrices are not invertible (they would be inverse to each other in the case of infinite matrices), although $\Lambda_+ \cdot \Lambda_- = 1$. The point is that the inverse order in this product does not lead to the unit matrix. By the same reason, condition (5.19) can not be replaced, say, by the similar condition $\Lambda_+ A = A \Lambda_+$ (or any of remaining two possibilities). This reflects the fact that only the first of eqs.(5.2) makes good sense in the forced RTC case.

Condition (5.19) means that $A_{ij} = A_{i-j}$ and

$$A(z,w) = \frac{\mu'(z)}{2\pi i z} \delta(z - w^{-1}) \tag{5.20}$$

where $\mu(z)$ is some arbitrary measure (compare with (3.1)).

Thus, RTC-reduction is described by Toeplitz matrices $A_{ij} = A_{i-j}$ which give rise to the corresponding subspace in the whole Grassmannian given by arbitrary matrices A_{ij}. In the next subsection, we demonstrate this by the direct calculation from the unitary matrix model representation of the RTC.

5.3 Determinant representation

One can show from formulas (5.14)-(5.18) that τ-function (5.11) of the forced hierarchy has the following determinant representation [22]

$$\tau_n(t) = \det \left[\partial_{t_1}^i (-\partial_{t_{-1}})^j \int_\gamma A(z,w) \exp\left\{ \sum_{m>0} (t_m z^m - t_{-m} w^{-m}) \right\} dz dw \right] \Bigg|_{i,j=0,\ldots,n-1} \tag{5.21}$$

Let us demonstrate now that the unitary matrix model (= forced RTC) leads to this determinant representation with A as in (5.20).

We begin with rewriting the orthogonality relation (3.2) in the matrix form. For doing this, we define matrices D and $D^\star$ with the matrix elements determined as the coefficients of the polynomials $\Phi_n(z)$ and $\Phi_n^\star(z)$

$$\Phi_i(z) \equiv \sum_j D_{ij} z^{j-1}, \quad \Phi_i^\star(z) \equiv \sum_j D_{ij}^\star z^{j-1} \tag{5.22}$$

Then, (3.2) looks like

$$D \cdot C \cdot D^{\star T} = H \tag{5.23}$$

where superscript T means transponed matrix and H denotes the diagonal matrix with the entries $C_{ii} = h_{i-1}$ and C is the moment matrix with the matrix elements

$$C_{ij} \equiv \int_\gamma \frac{d\mu(z)}{2\pi i z} z^{i-j} \exp\left\{ \sum_{m>0} (t_m z^m - t_{-m} z^{-m}) \right\} \tag{5.24}$$

Let us note that D ($D^{\star T}$) is the upper (lower) triangle matrix with the units on the diagonal (because of (3.4)). Indeed, this representation is nothing but the Riemann-Hilbert problem for the forced hierarchy. Now taking the determinant of the both sides of (5.23), one gets

$$\det_{n\times n} C_{ij} = \prod_{k=0}^{n-1} h_k = \tau_n \tag{5.25}$$

due to formula (3.3). The remaining last step is to observe that

$$C_{ij} = \partial_{t_1}^i (-\partial_{t_{-1}})^j \int_\gamma \frac{d\mu(z)}{2\pi i z} \exp\left\{ \sum_{m>0} (t_m z^m - t_{-m} z^{-m}) \right\} = \partial_{t_1}^i (-\partial_{t_{-1}})^j C_{11} \tag{5.26}$$

i.e.

$$\tau_n(t) = \det_{n\times n} \left[\partial_{t_1}^i (-\partial_{t_{-1}})^j \int_\gamma \frac{d\mu(z)}{2\pi i z} \exp\left\{ \sum_{m>0} (t_m z^m - t_{-m} z^{-m}) \right\} \right] \tag{5.27}$$

This expression coincides with (5.21) with $A(z,w)$ chosen as in (5.20). One can also remark that the moment matrix C_{ij} is Toeplitz matrix satisfying formula (5.19). This proves from the different approach that the RTC-reduction is defined by the Toeplitz matrices.

6 Relativistic Toda molecule

6.1 General properties

In this section we consider further restrictions on the RTC which allows one to consider the *both* products in (5.2). Namely, in addition to the condition (5.13) picking up forced hierarchy, we impose the following constraint

$$\tau_n = 0, \quad n > N \tag{6.1}$$

for some N. This system should be called $N-1$-particle relativistic Toda molecule, by analogy with the non-relativistic case and is nothing but RTC-reduction of the two-dimensional Toda molecule [23, 24][13].

Now we describe this system in different representations. Let us start with the general two-dimensional Toda molecule and discuss which element g in (5.10) describes this hierarchy. Since the Toda molecule is the very particular case of the forced

[13]Sometimes the Toda molecule is called non-periodic Toda [25]. It is an immediate generalization of the Liouville system.

hierarchy, we can look at representation (5.18). Then, the $sl(N)$ Toda can be described by the matrix A_{ij} of finite rank N [15]. This means that it can be presented as the finite sum

$$A_{ij} = \sum_{k}^{N} f_i^{(k)} g_j^{(k)} \tag{6.2}$$

where $f_i^{(k)}$ and $g_j^{(k)}$ are arbitrary coefficients. For the kernel $A(z,w)$ (5.18), this condition is

$$A(z,w) = \sum_{k}^{N} f^{(k)}(z) g^{(k)}(w) \tag{6.3}$$

where $f^{(k)}(z)$ and $g^{(k)}(z)$ are arbitrary functions. Indeed, these functions (or the sets of coefficients) describe the way how $sl(N)$ group can be embedded into $gl(\infty)$ (different embeddings are related by the external $gl(\infty)$ automorphisms of $gl(N)$) [15] – this is why the system is sometimes called $sl(N)$ Toda molecule. From this description, one can immediately read off the corresponding determinant representation (5.21).

Indeed, equation (3.28) and condition (6.1) implies that $\log \tau_0$ and $\log \tau_N$ satisfy the free wave equation

$$\partial_{t_1} \partial_{t_{-1}} \log \tau_0 = \partial_{t_1} \partial_{t_{-1}} \log \tau_N = 0 \tag{6.4}$$

Since the relative normalization of τ_n's is not fixed, we are free to choose $\tau_0 = 1$. Then,

$$\tau_0(t) = 1 \ , \qquad \tau_N(t) = \chi(t_1)\bar{\chi}(t_{-1}) \tag{6.5}$$

where $\chi(t_1)$ and $\bar{\chi}(t_{-1})$ are arbitrary functions. 2DTL with boundary conditions (6.5) was considered in [23]. The solution to (3.28) in this case is given by [24]:

$$\tau_n(t) \ = \ \det \ \partial_{t_1}^{i-1} (-\partial_{t_{-1}})^{j-1} \tau_1(t) \tag{6.6}$$

with

$$\tau_1(t) = \sum_{k=1}^{N} a^{(k)}(t) \bar{a}^{(k)}(t_{-1}) \tag{6.7}$$

where functions $a^{(k)}(t)$ and $\bar{a}^{(k)}(t_{-1})$ satisfy

$$\det \partial_{t_1}^{i-1} a^{(k)}(t) = \chi(t) \ , \ \ \det(-\partial_{t_{-1}})^{i-1} \bar{a}^{(k)}(t_{-1}) = \bar{\chi}(t_{-1}) \tag{6.8}$$

This result coincides with that obtained by substituting into (5.21) the kernel $A(z,w)$ of the form (6.3).

Let us stress that, although the Toda molecule looks as the forced hierarchy with one more projector inserted, this is described by the infinite number of the fermionic modes, since infinitely many matrix elements of (6.2) are not zero. However, the Toda molecule can be described by the finite matrix Lax representation how we shall see in the next subsection.

Now we have to describe the RTC-reduction of the Toda molecule. It is clear that, to do this, one needs to impose on the C_{11}-element of the moment matrix (5.26) to have the structure (6.6), or, which is the same, to require that the matrix $\partial_{t_1}^{i-1} (-\partial_{t_{-1}})^{j-1} \sum_{k=1}^{N} a^{(k)}(t) \bar{a}^{(k)}(t_{-1})$ to be Toeplitz. These conditions leads to some equations with the solution just giving the RTC Toda molecule, i.e. picking up the RTC-reduction among all the Toda molecule solutions. The same reduction also can be formulated as the condition for matrix (6.2) to be Toeplitz which looks quite tricky since naively leads to overfulled system of equations.

However, in the last subsection of this section we investigate the simplest $sl(2)$ case in detail and demonstrate that these conditions have the non-trivial solutions, i.e. the relativistic Toda molecule does really exist. Indeed, how is already clear from the next subsection, the RTC Toda molecule is the system with finite $(N-1)$ degrees of freedom, described by $N-1$ independent time flows and possessing, therefore, finitely many conservation laws. Unlike this, the whole 2DTL hierarchy, and its any reductions considered above (RTC, forced etc) are the systems with infinitely many degrees of freedom. The RTC molecule is so "small" because of rigidity of the two simultaneous reductions: the RTC and Toda molecule ones. In a sense, these two reductions are almost "perpendicular" remaining very small room for common solutions.

6.2 Lax representation

In all our previous considerations, we dealt with infinite-dimensional matrices. Let us note that the Toda molecule can be effectively treated in terms of $N \times N$ matrices like the forced case could be described by the quarter-infinite matrices. This allows one to deal with the *both* identities (5.2) since all the products of *finite* matrices are well-defined.

To see this, one can just look at the recurrent relation (3.58) and observe that there exists the finite-dimensional subsystem of (N) polynomials which is decoupled from the whole system. The recurrent relation for these polynomials can be considered as

the finite-matrix Lax operator (which still does not depend on the spectral parameter, in contrast to (2.22)). Indeed, from (3.58) and condition (6.1), i.e. $h_n/h_{n-1} = 0$ as $n \geq N$ (the Toda molecule conditions in terms of S-variables read as $S_n = S_n^\star = 1$ for $n > N-2$ or $n > 0$), one can see that

$$z\mathcal{P}_N(z) = \mathcal{P}_{N+1}(z) - \mathcal{P}_N(z), \quad z\mathcal{P}_{N+1}(z) = \mathcal{P}_{N+2}(z) - \mathcal{P}_{N+1}(z) \quad \text{etc.} \tag{6.9}$$

i.e. all the polynomials $\mathcal{P}_n$ with $n > N$ are trivially expressed through $\mathcal{P}_N$. Therefore, the system can be effectively described by the dynamics of only some first polynomials (i.e. has really finite number of degrees of freedom). Certainly, all the same is correct for the star-polynomials $\mathcal{P}_n^\star$ although, in this case, it would be better to use the original non-singularly normalized polynomials $\Phi_n^\star$.

Now let us look at the corresponding Lax operators (4.17)-(4.18). They are getting quite trivial everywhere but in the left upper corner of the size $N \times N$. For instance,

$$\mathcal{L} = \begin{pmatrix} -S_0 & 1 & & & \vdots & & & \\ -\frac{h_1}{h_0}S_1 & -S_1S_0^\star & 1 & & \vdots & & & \\ & \dots & & 1 & \vdots & & & \\ \dots & -\frac{h_{N-1}}{h_k}S_{k-1}^\star & \dots & -S_{N-2}^\star & \vdots & & & \\ \dots & \dots & \dots & \dots & -1 & 1 & & \\ & & & & & -1 & 1 & \\ & & & & & & -1 & 1 \\ & & & & & & & \ddots \end{pmatrix} \tag{6.10}$$

and analogously for the Lax operator $\bar{\mathcal{L}}$. Therefore, one can restrict himself to the system of N polynomials $\mathcal{P}_n$, $n = 0, 1, \dots, N-1$ and the finite matrix Lax operators (of the size $N \times N$).

Now one needs only to check that this finite system still has the same evolution equations (4.28). It turns out to be the case only for the first $N-1$ times. This is not so surprising, since, in the finite system with $N-1$ degrees of freedom, only first $N-1$ time flows are independent. Therefore, if looking at the finite matrix Lax operators, one gets the dependent higher flows. On the other hand, if one embeds this finite system into the infinite 2DTL, one observes that the higher flows can be no longer described inside this finite system. Let us remark that just the finite system is often called relativistic Toda molecule (see, for example, [16]).)

To simplify further notations, we introduce, instead of S_n, $S_n^\star$, the new variables $s_n \equiv (-)^{n+1}S_n$, $s_n^\star \equiv (-)^{n+1}S_n^\star$. Then, one can realize a very interesting property of the Lax operator (6.10) – it can be constructed as the product of simpler ones:

$$\mathcal{L} = \mathcal{L}_N\mathcal{L}_{N-1}\dots\mathcal{L}_1 \tag{6.11}$$

where $\mathcal{L}_k$ is the unit matrix wherever but a 2×2-block:

$$\mathcal{L}_k \equiv \begin{pmatrix} 1 & \vdots & \\ \cdots & G_k & \cdots \\ & \vdots & 1 \end{pmatrix} \qquad G_k \equiv \begin{pmatrix} s_k & 1 \\ s_k s_k^\star - 1 & s_k^\star \end{pmatrix} \tag{6.12}$$

Analogously

$$\bar{\mathcal{L}} = \bar{\mathcal{L}}_1\dots\bar{\mathcal{L}}_{N-1}\bar{\mathcal{L}}_N \tag{6.13}$$

with

$$\bar{\mathcal{L}}_k \equiv \begin{pmatrix} 1 & \vdots & \\ \cdots & \bar{G}_k & \cdots \\ & \vdots & 1 \end{pmatrix} \qquad \bar{G}_k \equiv \begin{pmatrix} s_k^\star & -1 \\ 1 - s_k s_k^\star & s_k \end{pmatrix} \tag{6.14}$$

One can trivially see that $\mathcal{L}_k\bar{\mathcal{L}}_k = \bar{\mathcal{L}}_k\mathcal{L}_k = 1$, and, therefore, from (6.11) and (6.13), one obtains $\mathcal{L}\bar{\mathcal{L}} = \bar{\mathcal{L}}\mathcal{L} = 1$ (cf. (5.2)).

From formulas (6.11)-(6.13), one trivially obtains that $\det \mathcal{L} = \det \bar{\mathcal{L}} = 1$ which reminds once more of the $sl(N)$ algebra. More generally, the factorization property of the Lax operators opens the wide road to the group theory interpretation of the RTC molecule. Indeed, following the line of papers [14, 15, 16], one should identify the (family of) solutions to the integrable hierarchies with representations of the algebra of functions on the underlying group manifold. In the quantum group case, one should expect that different reductions correspond to fixing the irreducible representations [14, 15]. As the classical counterpart of this statement, one should consider the (quadratic) Poisson structure for the group elements[14]

$$\{g \underset{,}{\otimes} g\} = [r, g \otimes g] \tag{6.15}$$

where r is the classical r-matrix (see, e.g., [8]), and, instead of irreducible representations, look at the simplectic leaves, i.e. the submanifolds where Poisson structure is non-degenerate (an analog of the geometric quantization) – see [17]. This procedure has been performed in [16] for the $sl(N)$ group and was shown to lead to the relativistic Toda molecule for the leaves of the dimension equal to the doubled rank of the group (i.e. $2(N-1)$ for $sl(N)$).

It was shown in [17] that there always exist such leaves[15] and they were manifestly described. Namely, for the $sl(2)$ case, the 2-dimensional simplectic leaf can be described by the group element of the form

$$g = \begin{pmatrix} a & b \\ b & d \end{pmatrix} \qquad ad - b^2 = 1 \tag{6.16}$$

Multiplying this matrix by the trivial rescaling matrix $g = \begin{pmatrix} b^{-1} & \\ & b \end{pmatrix}$ one can easily transform g to the matrix G_k in (6.12).

In fact, the only important information of this matrix at the moment is that it is parametrized by a 2-dimensional manifold. However, the crucial test comes when considering the structure of this matrix for higher rank groups. Namely, for arbitrary $sl(N)$ group, the group element corresponding to the proper simplectic leaf is constructed from elementary $sl(2)$ building blocks exactly how it is done in formula (6.11) [17, 16]. Therefore, the Lax operator of the relativistic $sl(N)$ Toda molecule can be treated as the group element of the special form corresponding to the simplectic leaf of the dimension $2(N-1)$. Then, one can construct the Hamiltonians commuting with respect to the Poisson structure (6.15) and giving the time flows just as traces of the Lax operator.

It may seem that there does not remain any room for the other Lax operator (6.13). However, this is not the case. Indeed, the above construction is correct up to any Weyl transformation. The transformation corresponding to the longest element of the Weyl group maps $\mathcal{L}$ operator to $\bar{\mathcal{L}}$ operator. This explains the symmetry of dynamics with respect to the positive and negative times. As for other Weyl transformations, the corresponding Hamiltonians can be hardly into 2DTL dynamics.

6.3 $sl(2)$ molecule

To illustrate the results of this section, let us consider the simplest $sl(2)$ example of the relativistic Toda molecule. This case is to be considered as a "relativization" of the Liouville theory. Surprisingly enough, this case turns out to be *equivalent* to the Liouville theory. This is a specific feature of the $sl(2)$ case, which is no longer correct already for the $sl(3)$ group. This can be demonstrated by comparing the Lax operators, but we use more immediate way of comparing solutions to the equations of hierarchy.

In fact, the $sl(2)$ hierarchy corresponding to the rank one group contains only two equations: one for positive and the other one for negative times. One of these equations is (1.1) with the Toda molecule constraints $q_{-1} \to \infty$, $q_2 \to -\infty$ etc. Under these constraints, there are only two nontrivial RTC equations

$$\begin{aligned} \ddot{q}_0 &= (1+\epsilon\dot{q}_0)(1+\epsilon\dot{q}_1)\frac{e^{q_1-q_0}}{1+\epsilon^2 e^{q_1-q_0}} \\ \ddot{q}_1 &= -(1+\epsilon\dot{q}_0)(1+\epsilon\dot{q}_1)\frac{e^{q_1-q_0}}{1+\epsilon^2 e^{q_1-q_0}} \end{aligned} \tag{6.17}$$

These equations imply that $q_0 + q_1 = C_1 t + C_2$ with arbitrary constants $C_1,\ C_2$. Now introducing the new variable $q \equiv q_0 - \frac{1}{2}(C_1 t + C_2)$ one gets the equation

$$\ddot{q} = \left[(1+\frac{1}{2}\epsilon C_1)^2 - \epsilon^2\dot{q}^2\right]\frac{e^{-2q}}{1+\epsilon^2 e^{-2q}} \tag{6.18}$$

[14]In paper [26], the same ideology was applied to the linear Poisson structures. This leads to the non-relativistic TC, in accordance with the "algebraic" character of this latter as opposed to the "group" character of the RTC.

[15]When quantizing, they lead to the infinite-dimensional representations of the algebra of functions.

The limit of $\epsilon \to 0$ leads to the Liouville equation

$$\ddot{q} = e^{-2q} \tag{6.19}$$

Equation (6.18) has the solution

$$\int \frac{dq}{\sqrt{\frac{(1+\frac{1}{2}\epsilon C_1)^2+C_3}{\epsilon^2} + C_3 e^{-2q}}} = t_1 + C_4 \tag{6.20}$$

$$\text{i.e.} \quad q = \log\left[A_0 \sinh(A_1 t_1 + A_2)\right]$$

(with some redefined constants $\{C_i\} \to \{A_i\}$) which is to be compared with the analogous solution to the Liouville equation (non-relativistic Toda molecule)

$$\int \frac{dq}{\sqrt{C_3 - e^{-2q}}} = t_1 + C_4 \tag{6.21}$$

One easily observes that these both can be transformed to each other by the trivial rescaling of the time variable and integration constants. Therefore, "the relativization" of the Liouville equation does not lead to a new equation.

Let us note, however, that switching on the negative time changes this statement. Indeed, looking at equation completely equivalent to (1.1), one can check that the solutions really depend on an arbitrary linear combination of times:

$$q = \log\left[A_0 \sinh(A_1 t_1 + A_3 t_{-1} + A_2)\right] \tag{6.22}$$

Therefore, the notorious property of dependence only on the sum of positive and negative times, which distinguishes the TC hierarchy is not fulfilled for the RTC molecules. Thus, these systems are really different even in the $sl(2)$ case.

In fact, the pair of equations (6.17) can be really substituted by the only equation for the τ-function τ_1 (3.28) with the Toda molecule (6.5) and reduction conditions. To do this, let us first note that conditions (6.4) allows one to normalize τ-functions so that $\tau_0 = \tau_N = 1$. In particular, in the $sl(2)$ case this means that one can add to the equation (3.28) for τ_1 analogous equations for τ_0 and τ_2 with the coefficient $-\frac{1}{2}$ to get

$$\frac{1}{2}\partial_{t_1}\partial_{t_{-1}} \log \frac{\tau_1^2}{\tau_0\tau_2} = -\frac{\tau_0\tau_2}{\tau_1^2} \tag{6.23}$$

and, for the new function $\tau \equiv \frac{\tau_1}{\sqrt{\tau_0\tau_2}}$,

$$\partial_{t_1}\partial_{t_{-1}} \log \tau = -\frac{1}{\tau^2} \tag{6.24}$$

Using formulas (3.9) and (3.13), one obtains that

$$\epsilon^2 e^{q_1 - q_0} = -\frac{\tau_0\tau_2}{\tau_1^2} = -\frac{1}{\tau^2} = \epsilon^2 e^{-2q} \tag{6.25}$$

We have already remarked (see the very end of sect.3.1) that, generically, the connection of τ-functions with q-variables may require some additional shift of these latter.

Now one should solve equation (6.24) with the moment matrix $\partial_{t_1}^{i-1}(-\partial_{t_{-1}})^{j-1}\tau_1(t)$ constrained to be Toeplitz. In our case, this means that $\partial_{t_1}\partial_{t_{-1}}\tau = -\tau$. Then, one obtains the equation (the same equation can be obtained from (6.18) by the substitute (6.25))

$$\tau = \frac{\tau^2 - 1}{(\partial_{t_1}\tau)^2}\partial_{t_1}^2\tau \tag{6.26}$$

This equation has the solution

$$\int \frac{d\tau}{\sqrt{\tau^2+1}} = B_1 t_1 + B_2 \tag{6.27}$$

i.e.

$$\tau = \sinh(B_1 t_1 + B_2) \tag{6.28}$$

and can be reduced to (6.20) through the replace (6.25) $\tau = \frac{i}{\epsilon}e^{q+\text{const}}$. This formula differs from (6.25) by a constant shift of q. Indeed, we have already remarked (see the very end of sect.3.1) that, generically, the connection of τ-functions with q-variables may require some additional shift of these latter. In the present case, the shift depends on value of C_1 and is in charge of the non-unit coefficient A_0 in (6.22). The correct connection of τ and q restores if C_1 is chosen to be $-\frac{2}{\epsilon}$, i.e. $q_n \to q_n - \frac{1}{\epsilon}t_1$.

Now let us look at the dependence of τ on the negative time. Naively, using (6.28) and constraint $\partial_{t_1}\partial_{t_{-1}}\tau = -\tau$, one gets that the general solution is given by the formula

$$\tau = \sinh(B_1 t_1 - B_1^{-1} t_{-1} + B_2) \tag{6.29}$$

which contradicts to (6.22) with arbitrary unrelated coefficients in front of times. This puzzle is solved by noting that the moment matrix (5.26) can be multiplied by arbitrary triangle matrix with units on the diagonal so that the determinant (5.27)

does not change. This means that one really needs to consider the weaker Toeplitz condition $\partial_{t_1}\partial_{t_{-1}}\tau + \alpha\partial_{t_{-1}}\tau = -\tau$, where α is a constant (this does not effect equation (6.26)). This results to the general solution for τ-function

$$\tau = \sinh(B_1 t_1 + B_3^{-1} t_{-1} + B_2) \tag{6.30}$$

consistent with (6.22). This provides us with some nontrivial example of the simultaneous solutions to the constraints of the relativistic Toda molecule and Toeplitz moment matrix. The measure $\mu'(z)$ (5.20) can be obtained from the explicit formulas (6.28) and (5.27) by the Fourier transform.

7 Discrete evolutions and limit to Toda chain

7.1 Darboux-Bäcklund transformations

In this section, we are going to discuss some discrete evolutions of the RTC given by the Darboux-Bäcklund transformations and their limit to the usual Toda chain. One can easily take the continuum limit of the formulas of this section to reproduce the TC as the limit of the RTC, both with the standard continuous evolutions.

The discrete evolution equations in the RTC framework were recently introduced by [3] in a little bit sophisticated way. Here we outline the simple approach based on the notion of the Darboux-Bäcklund transformation (DBT). More details will be presented in the separate publication [19].

Let discrete index i denote the successive DBT's. The spectral problem now can be written as follows:

$$\Phi_{n+1}(i|z) + a_n(i)\Phi_n(i|z) = z\Big\{\Phi_n(i|z) + b_n(i)\Phi_{n-1}(i|z)\Big\} \tag{7.1}$$

In fact, we consider two pairs of different DBT's, the two in each pair being complimentary to each other. This is why we call the DBT's in a pair forward and backward DBT's.

Let us define the first forward DBT (treating it as the discrete evolution) in the form very similar to that of the usual Toda chain:

$$\Phi_n(i+1|z) = \Phi_n(i|z) + \boldsymbol{\alpha}_n^{(1)}(i)\Phi_{n-1}(i|z) \tag{7.2}$$

where $\boldsymbol{\alpha}_n^{(1)}(i)$ are some unknown functions. From the point of view of the whole hierarchy, this new evolution means nothing but a discretized version of the first negative flow equation (2.2). One requires that $\Phi_n(i+1)$ satisfies the same spectral problem as (7.1) but with the shifted value of i:

$$\Phi_{n+1}(i+1|z) + a_n(i+1)\Phi_n(i+1|z) = z\Big\{\Phi_n(i+1|z) + b_n(i+1)\Phi_{n-1}(i+1|z)\Big\} \tag{7.3}$$

Then the compatibility condition gives the equations of the discrete RTC:

$$a_n(i+1) = a_{n-1}(i)\,\frac{a_n(i) - \boldsymbol{\alpha}_{n+1}^{(1)}(i)}{a_{n-1}(i) - \boldsymbol{\alpha}_n^{(1)}(i)} \tag{7.4}$$

$$b_n(i+1) = b_{n-1}(i)\,\frac{b_n(i) - \boldsymbol{\alpha}_n^{(1)}(i)}{b_{n-1}(i) - \boldsymbol{\alpha}_{n-1}^{(1)}(i)} \tag{7.5}$$

$$z_i\frac{b_n(i)}{\boldsymbol{\alpha}_n^{(1)}(i)} = z_i - a_n(i) + \boldsymbol{\alpha}_{n+1}^{(1)}(i) \tag{7.6}$$

where z_i are arbitrary constants. These equations give some natural generalization of the corresponding discrete evolution for the Toda chain hierarchy; moreover, there exists the simple limit to the usual discrete Toda equations (see the discussion of this limit below). Using representation (2.6), (2.7) it is very easy to rewrite the system (7.4)-(7.6) in terms of "coordinates" $q_n(i)$ only [3].

From the continuum picture, one can get that there is another simple evolution (t_1-flow) treated in a parallel way with that generated by the t_{-1}-flow. Therefore, one can try to find the corresponding discrete analog of (2.16). Indeed, such analog exists and has the natural form

$$\Phi_n(i+1|z) = \Big(1 - \boldsymbol{\alpha}_n^{(2)}(i)\Big)\Phi_n(i|z) + z\boldsymbol{\alpha}_n^{(2)}(i)\Phi_{n-1}(i|z) \tag{7.7}$$

where $\boldsymbol{\alpha}_n^{(2)}(i)$ are some new unknown functions of the corresponding discrete indices. We refer to this evolution as to the second forward DBT. Substitution of (7.7) to (7.1) gives quite different system of the discrete evolution equations:

$$a_n(i+1) = a_n(i)\frac{1 - \boldsymbol{\alpha}_{n+1}^{(2)}(i)}{1 - \boldsymbol{\alpha}_n^{(2)}(i)}) \tag{7.8}$$

$$b_n(i+1) = b_{n-1}(i)\frac{\boldsymbol{\alpha}_n^{(2)}(i)}{\boldsymbol{\alpha}_{n-1}^{(2)}(i)} \tag{7.9}$$

$$a_n(i) + b_n(i)\ \frac{1-\boldsymbol{\alpha}_n^{(2)}(i)}{\boldsymbol{\alpha}_n^{(2)}(i)} = z_i\ \frac{1}{1-\boldsymbol{\alpha}_{n+1}^{(2)}(i)} \tag{7.10}$$

This system is the discrete counterpart of the continuum system (2.17), (2.18). Written in the terms of $q_n(i)$, equations (7.8)-(7.10) define the same evolution of coordinates as in the case of the first forward DBT.

Actually, in [3], four different discrete systems of the RTC equations were written. From our point of view, the additional evolutional systems result from the **backward Darboux-Bäcklund transformations** which are complimentary to those described above. For example, the first backward DBT has the form

$$\Phi_n(i|z) = \frac{1}{z-z_i}\Big\{\Big(1-\boldsymbol{\beta}_n^{(1)}(i)\Big)\Phi_{n+1}(i+1|z) + z\boldsymbol{\beta}_n^{(1)}(i)\Phi_n(i+1|z)\Big\} \tag{7.11}$$

where z_i are the same constants as in (7.6) and variables $\boldsymbol{\beta}_n^{(1)}(i)$ are related with the variables of the first forward DBT as follows:

$$\boldsymbol{\beta}_n^{(1)}(i) = \frac{b_n(i)}{b_n(i)-\boldsymbol{\alpha}_n^{(1)}(i)} \tag{7.12}$$

The system obtained from (7.11) can be easily reduced to (7.4)-(7.6) and is not considered here anymore.

7.2 Continuum limit

Introducing some discrete shift of time $\Delta > 0$, one can rewrite all the equations describing the first forward DBT as follows:

$$\Phi_n(t+\Delta) = \Phi_n(t) + \boldsymbol{\alpha}_n^{(1)}(t)\Phi_{n-1}(t) \tag{7.13}$$

$$a_n(t+\Delta) = a_{n-1}(t)\ \frac{a_n(t)-\boldsymbol{\alpha}_{n+1}^{(1)}(t)}{a_{n-1}(t)-\boldsymbol{\alpha}_n^{(1)}(t)} \tag{7.14}$$

$$b_n(t+\Delta) = b_{n-1}(t)\ \frac{b_n(t)-\boldsymbol{\alpha}_n^{(1)}(t)}{b_{n-1}(t)-\boldsymbol{\alpha}_{n-1}^{(1)}(t)} \tag{7.15}$$

In order to get the proper limit in (7.6) one should rescale the constants z_i ; for simplicity, we assume now that they do not depend on i :

$$z_i \to g\ \Delta \tag{7.16}$$

Then we get

$$g\Delta\ \frac{b_n(t)}{\boldsymbol{\alpha}_n^{(1)}(t)} = g\ \Delta - a_n(t) + \boldsymbol{\alpha}_{n+1}^{(1)}(t) \tag{7.17}$$

and solution to (7.17) has the asymptotics

$$\boldsymbol{\alpha}_n^{(1)}(t) = -g\Delta\ \frac{b_n(t)}{a_n(t)}\ +\ O(\Delta^2) \tag{7.18}$$

and, therefore, from (7.13)-(7.15)

$$\frac{1}{\Delta}\Big\{\Phi_n(z,t+\Delta) - \Phi_n(z,t)\Big\}\ =\ -\ g\ \frac{b_n(t)}{a_n(t)}\Phi_{n-1}(z,t)\ +\ O(\Delta) \tag{7.19}$$

$$\frac{1}{\Delta}\Big\{a_n(t+\Delta) - a_n(t)\Big\}\ =\ -\ g\ \left(\frac{b_n(t)}{a_{n-1}(t)} - \frac{b_{n+1}(t)}{a_{n+1}(t)}\right)\ +\ O(\Delta) \tag{7.20}$$

$$\frac{1}{\Delta}\Big\{b_n(t+\Delta) - b_n(t)\Big\}\ =\ -\ g\ b_n(t)\left(\frac{1}{a_{n-1}(t)} - \frac{1}{a_n(t)}\right)\ +\ O(\Delta) \tag{7.21}$$

and, in the continuum limit, $\Delta \to 0$ these equations lead exactly to (2.4), (2.5).

The analogous equations can be written for the second forward DBT but now z_i should be rescaled as follows:

$$z_i \to \frac{1}{g\ \Delta} \tag{7.22}$$

and (7.7), (7.8) - (7.10) now have the form

$$\Phi_n(t+\Delta) = \Big(1-\boldsymbol{\alpha}_n^{(2)}(t)\Big)\Phi_n(t) + z\boldsymbol{\alpha}_n^{(2)}(t)\Phi_{n-1}(t) \tag{7.23}$$

Substituting to (7.1) gives the equations:

$$a_n(t+\Delta) = a_n(t)\frac{1-\boldsymbol{\alpha}_{n+1}^{(2)}(t)}{1-\boldsymbol{\alpha}_n^{(2)}(t)}) \tag{7.24}$$

$$b_n(t+\Delta) = b_{n-1}(t)\frac{\boldsymbol{\alpha}_n^{(2)}(t)}{\boldsymbol{\alpha}_{n-1}^{(2)}(t)} \tag{7.25}$$

$$a_n(t) + b_n(t)\ \frac{1-\boldsymbol{\alpha}_n^{(2)}(t)}{\boldsymbol{\alpha}_n^{(2)}(t)} = \frac{1}{g\ \Delta}\ \frac{1}{1-\boldsymbol{\alpha}_{n+1}^{(2)}(t)} \tag{7.26}$$

From (7.26), it follows that

$$\boldsymbol{\alpha}_n^{(2)}(t) \simeq g\Delta\ b_n(t)\Big\{1 + g\Delta\ \Big(a_n(t) - b_n(t) - b_{n+1}(t)\Big)\Big\} \tag{7.27}$$

and, therefore,

$$\frac{1}{\Delta}\Big\{\Phi_n(z,t+\Delta) - \Phi_n(z,t)\Big\}\ =\ -\ g\ b_n(t)\Big(\Phi_n(z,t) - z\Phi_{n-1}(z,t)\Big)\ +\ O(\Delta) \tag{7.28}$$

$$\frac{1}{\Delta}\Big\{a_n(t+\Delta) - a_n(t)\Big\}\ =\ g\ a_n(t)\Big(b_n(t) - b_{n+1}(t)\Big)\ +\ O(\Delta) \tag{7.29}$$

$$\frac{1}{\Delta}\Big\{b_n(t+\Delta) - b_n(t)\Big\}\ =\ g\ b_n(t)\Big(a_n(t) - a_{n-1}(t) + b_{n-1}(t) - b_{n+1}(t)\Big)\ +\ O(\Delta) \tag{7.30}$$

It is clear that, in the limit $\Delta \to 0$, one reproduces the continuum equations (2.17), (2.18).

7.3 Limit to Toda chain

Here we very briefly outline the theory of the discrete evolution of the usual Toda chain (see, for example, [27] and references therein). The spectral problem for the Toda chain has the form

$$\lambda\Psi_n(i|\lambda) = \Psi_{n+1}(i|\lambda) - p_n(i)\Psi_n(i|\lambda) + R_n(i)\Psi_{n-1}(i|\lambda) \tag{7.31}$$

where we consider the whole set of successive DBT labelled by the discrete index i as a discretized variant of the continuous evolution equations

$$\delta\Psi_n(i) \equiv \Psi_n(i+1) - \Psi_n(i) = A_n(i)\Psi_{n-1}(i) \tag{7.32}$$

which should be compatible with the spectral problem (7.31). (7.32) describes the forward DBT. Analogously, the backward DBT is

$$\Psi_n(i|\lambda) = \frac{1}{\lambda-\lambda_i}\Big\{\Psi_{n+1}(i+1|\lambda) + B_n(i)\Psi_n(i+1|\lambda)\Big\} \tag{7.33}$$

where coefficients $B_n(i)$ should be determined. The compatibility conditions give the following system of the evolution equations:

$$p_n(i) = -\lambda_i - A_{n+1}(i) - B_n(i) \tag{7.34}$$

$$p_n(i+1) = -\lambda_i - A_n(i) - B_n(i) \tag{7.35}$$

$$R_n(i) = A_n(i)B_n(i) \tag{7.36}$$

$$R_n(i+1) = A_n(i)B_{n-1}(i) \tag{7.37}$$

where λ_i serve as free parameters. In order to get the close connection with the general theory of integrable systems as well as with the theory of orthogonal polynomials, one can introduce new variables $h_n(i) \equiv \exp\Big(q_n(i)\Big)$ through the relation:

$$R_n(i)\ \equiv\ \frac{h_n(i)}{h_{n-1}(i)} \tag{7.38}$$

Then, in the terms of variables $h_n(i)$

$$A_n(i) = \frac{h_n(i+1)}{h_{n-1}(i)} \tag{7.39}$$

$$B_n(i) = \frac{h_n(i)}{h_n(i+1)} \tag{7.40}$$

and, from (7.34)-(7.35), one can get the following evolution equations for the discrete Toda chain (assuming for simplicity that parameters λ_i do no depend on i):

$$\frac{h_n(i+1)h_n(i-1)}{h_n^2(i)} = \frac{1-\dfrac{h_n(i+1)}{h_{n-1}(i-1)}}{1-\dfrac{h_{n+1}(i+1)}{h_n(i-1)}} \tag{7.41}$$

Now we can consider the transition from the RTC to the usual discrete-time Toda chain. Let (compare with (2.6), (2.7))

$$\begin{gathered} a_n(i) \simeq 1 - \epsilon p_n(i)\ \ ;\quad b_n(i) \simeq -\epsilon^2 R_n(i) \\ z \simeq 1 + \epsilon\lambda\ \ ;\quad z_i \simeq 1 + \epsilon\lambda_i \end{gathered} \tag{7.42}$$

Introduce also functions $\Psi_n(i)$[16]

$$\Phi_n(i) \simeq \epsilon^n \Psi_n(i) \tag{7.43}$$

It is easy to see that (7.2) leads to the forward DBT for the Toda chain if one identifies

$$\boldsymbol{\alpha}_n^{(1)}(i) \simeq \epsilon A_n(i) \tag{7.44}$$

In this case, for example, equation (7.6) reduces to (7.34). Actually, it is easy to see that the whole system (7.4)-(7.6) reduces to (7.34)-(7.37) in the limit $\epsilon \to 0$. Moreover, from (7.12) one gets

$$\boldsymbol{\beta}_n^{(1)}(i) = \epsilon \frac{R_n(i)}{A_n(i)} + O(\epsilon^2) \ \simeq \epsilon B_n(i) \tag{7.45}$$

thus obtaining the proper limit of the first backward DBT of the RTC (7.11) to that of the Toda chain (7.33).

It is clear also that, in this limit, the second forward DBT (7.7) reduces to (7.32) and all the equations we get from (7.8)-(7.10) are equivalent to (7.34)-(7.37) when $\epsilon \to 0$.

There exist some other interesting limits. In particular, one can consider the degenerate case of evolution equations (7.4)-(7.6) in the limit $z_i \to \infty$. In this case, the solution to (7.6) has the asymptotics

$$\boldsymbol{\alpha}_n^{(1)}(i) = b_n(i) + \frac{1}{z_i} b_n(i)\Big(a_n(i) - b_{n+1}(i)\Big) + O\Big(\frac{1}{z_i^2}\Big) \tag{7.46}$$

Thus, in this limit equations (7.4) and (7.5) take the simple form:

$$a_n(i+1) = a_{n-1}(i)\ \frac{a_n(i) - b_{n+1}(i)}{a_{n-1}(i) - b_n(i)} \tag{7.47}$$

$$b_n(i+1) = b_n(i)\ \frac{b_{n+1}(i) - a_n(i)}{b_n(i) - a_{n-1}(i)} \tag{7.48}$$

We should stress that these equations can be easily derived from the spectral problem (7.1) just representing it in the form

$$\Phi_{n+1}(i|z) + a_n(i)\Phi_n(i|z) = z\ \Phi_n(i+1|z) \tag{7.49}$$

$$\Phi_n(i+1|z) = \Phi_n(i|z) + b_n(i)\Phi_{n-1}(i|z) \tag{7.50}$$

Using representation (2.6), (2.7) we can rewrite system (7.47), (7.48) in terms of coordinates $q_n(i)$ as follows:

$$\exp\Big\{q_n(i+1) - 2q_n(i) + q_n(i-1)\Big\} = \frac{1 + \epsilon^2 \exp\Big\{q_{n+1}(i) - q_n(i)\Big\}}{1 + \epsilon^2 \exp\Big\{q_n(i) - q_{n-1}(i)\Big\}} \tag{7.51}$$

thus getting the counterpart of the discrete-time Toda chain [3]. We should stress, however, that equations (7.41) and (7.51) are obtained as different limits from the general discrete RTC (7.4)-(7.6). Moreover, they are described by different Lax operators.

8 Some applications to biorthogonal polynomials

In this section we discuss some peculiar properties of the system of (relativistic) polynomials leading to the RTC, biorthogonal with some specific measures.

8.1 Finite systems of orthogonal polynomials

We already discussed in sect.3 that the relativistic polynomials satisfy recurrent relations (3.5)

$$\begin{gathered}\Phi_{n+1}(z) = z\Phi_n(z) + S_n z^n \Phi_n^\star(z^{-1}) \\ \Phi_{n+1}^\star(z^{-1}) = z^{-1}\Phi_n^\star(z^{-1}) + S_n^\star z^{-n}\Phi_n(z)\end{gathered} \tag{8.1}$$

provided the polynomials are normalized to be the n-th order monic (i.e. with the coefficient in front of the leading term equal to 1) polynomials. In particular, the initial conditions are $\Phi_0(z) = \Phi_0^\star(z) = 1$. We also demonstrated that such polynomials satisfy the RTC recurrent relations (2.1)

$$\Phi_{n+1}(z) + a_n\Phi_n(z) = z\{\Phi_n(z) + b_n\Phi_{n-1}(z)\}\ , \quad n \in \mathbb{Z} \tag{8.2}$$

[16]Let us note that, under this limit, the corresponding biorthogonal polynomials turn into the usual Toda orthogonal polynomials, with the biorthogonality relation (3.2) transforming to the orthogonality relation. This is why we sometimes call these biorthogonal polynomials relativistic ones.

As far as we know, it was G.Baxter who first considered the system (3.5) [28] (for recent development of the theory of these polynomials see e.g. [29, 30, 31]). On the other hand, Pastro showed [29] that the theory of the polynomials defined by (3.5) is closely related to the Laurent orthogonal polynomials first introduced in the framework of the strong Stiltjes moment problem [32]. It is interesting to note that the RTC was studied in [33] by the method of continued T-fractions arising in the context of the same strong Stiltjes moment problem, however, without using the orthogonal polynomials.

Note that in general, S_n and $S_n^\star$ are arbitrary complex parameters. When $S_n = \bar{S}_n^\star$ and $|S_n| < 1$, one gets the theory of Szegö polynomials orthogonal on the unit circle [34, 35]. The case of $|S_n| > 1$ is much less trivial and has not been considered in detail.

It is interesting to discuss how given recurrent relations ((3.5) or (2.1)) are connected to the concrete orthogonality measure. In general, when the space of the polynomials is infinite-dimensional, one can hardly formulate any statement on the existence and support of the corresponding measure. However, for the finite-dimensional representations corresponding to the Toda molecule such (discrete) measures can be found explicitly.

For doing this, note that one can easily derive from (3.5) the identity

$$y^{-n}(\tilde{\Phi}_n^\star(y)\Phi_{n+1}(x) - \tilde{\Phi}_n^\star(x)\Phi_{n+1}(y))h_n^{-1} = (x-y)\sum_{k=0}^{n} \Phi_k(x)\Phi_k^\star(y^{-1})h_k^{-1} \tag{8.3}$$

where $h_0 = 1$ and we introduced for the sake of brevity $\tilde{\Phi}_n^\star(z) \equiv z^n \Phi_n^\star(1/z)$. This identity is valid for arbitrary complex x and y unless $x \neq y$. For $x = y$ we have the identity

$$(y^n h_n)^{-1}(\tilde{\Phi}_n^\star(y)\Phi'_{n+1}(y) - \Phi_{n+1}(y)\tilde{\Phi}_n^{\star\prime}(y)) = \sum_{k=0}^{n} \Phi_k(y)\Phi_k^\star(1/y)h_k^{-1} \tag{8.4}$$

Identities (8.3) and (8.4) can be considered as "relativistic" analogs of the well known Darboux-Christoffel formula in the theory of the ordinary orthogonal polynomials [34].

Now let us turn to the finite set of polynomials. In order to reduce our system, we put $h_{n+1} = 0, \quad h_n \neq 0$ (this means that $S_n S_n^\star = 1$ and this corresponds to the Toda molecule case of section 6). Let $y_j, \quad j = 0, 1, \dots, n$ be the roots of the polynomial $\Phi_{n+1}(y)$. We assume additionally that all these roots are simple. Then, from (8.3) and (8.4), one gets the following orthogonality relation

$$\sum_{k=0}^{n} \Phi_k(y_s)\Phi_k^\star(y_r^{-1})h_k^{-1} = w_s^{-1}\delta_{rs} \tag{8.5}$$

where

$$w_s = \frac{h_n}{\Phi_n^\star(y_s^{-1})\Phi'_{n+1}(y_s)} \tag{8.6}$$

At last, using that, for the orthogonal matrices, their transponed are orthogonal too, one finally obtains the dual orthogonality relation

$$\sum_{k=0}^{n} w_k \Phi_n(y_k)\Phi_m^\star(y_k^{-1}) = h_n \delta_{nm} \tag{8.7}$$

Therefore, w_k is the discrete weight function whose support is the set of zeros of the polynomial $\Phi_{n+1}(z)$.

Note that because of the condition $S_n S_n^\star = 1$, the roots y_k of the polynomial $\Phi_{n+1}(z)$ coincide with the roots of the polynomial $\Phi_{n+1}^\star(1/z)$. Then, using the symmetry between the polynomials $\Phi_n(z)$ and $\Phi_n^\star(z)$, one derives another orthogonality relation

$$\sum_{k=0}^{n} \tilde{w}_k \Phi_n(y_k^{-1})\Phi_m^\star(y_k) = h_n \delta_{nm} \tag{8.8}$$

where

$$\tilde{w}_k = \frac{h_n}{\Phi_n(y_k)\Phi_{n+1}^{\star\prime}(y_k^{-1})} \tag{8.9}$$

In general, the weight functions w_k and $\tilde{w}_k$ do not coincide and, therefore, we have two different orthogonality relations for the same polynomials. However, if $S_k = S_k^\star, \quad k = 0, 1, \dots, n$ (i.e. we deal with the symmetric case) then $w_k = \tilde{w}_k$. Moreover, in this case the polynomials $\Phi_k(z)$ and $\Phi_k^\star(z)$ coincide and, hence, the set of the roots y_k of the polynomial Φ_{n+1} coincides with the set y_k^{-1} of roots of the polynomial $\Phi_{n+1}^\star(z)$. This means that, in the symmetric case, the polynomial $\Phi_{n+1}(z)$ is invertible, i.e. the coefficients in front of the terms z^j and z^{n+1-j} coincide (or, equivalently, all the roots enter with the pairs $y_k, \quad y_k^{-1}$). In particular, when $|S_k| < 1$, all the roots lie on the unit circle.

8.2 Anzatz of separated variables

Now let us consider another interesting peculiar case of the relativistic polynomials connected with a specific solution to the RTC hierarchy. Namely, look at the following ansatz of separated variables (see (2.1)) (for the non-relativistic Toda chain this ansatz was studied in [36])

$$a_n(t) = \gamma_1(t)\nu_n, \quad b_n(t) = \gamma_2(t)\mu_n \tag{8.10}$$

where $\gamma_i(t)$ depend on t only whereas ν_n and μ_n depend only on n. Putting $b_0 = 0$ (because of (3.4)), one gets the solution

$$a_n(t) = -\frac{t}{n+c+1}, \quad b_n(t) = -\frac{tn}{(n+c)(n+c+1)} \tag{8.11}$$

where a is an arbitrary constant. Then, the corresponding coefficients S_n and $S_n^\star$ are equal to

$$\alpha_n(t) = -\frac{t^n}{(c+1)_{n+1}}, \quad \beta_n(t) = -t^{-n}(c)_{n+1} \tag{8.12}$$

where $(c)_n = c(c+1)\dots(c+n-1)$ is the Pochhammer symbol.

In order to find explicit expressions for the polynomials $\Phi_n(z)$ and $\Phi_n^\star(z)$, note that the recurrent coefficients (8.12) or (8.11) can be obtained as some limiting case of the system of biorthogonal polynomials on the unit circle proposed by R.Askey [18]. Indeed, the Askey polynomials have the recurrent coefficients

$$S_n = -\frac{(b)_{n+1}}{(c+1)_{n+1}}, \quad S_n^\star = -\frac{(c)_{n+1}}{(b+1)_{n+1}} \tag{8.13}$$

where a, b are arbitrary parameters. Corresponding polynomials are expressed in terms of the Gauss hypergeometric function [18]

$$\Phi_n(z) = \frac{(b)_n}{(c+1)_n}\, {}_2F_1\left(\begin{matrix}-n, 1+c\\ 1-n-b\end{matrix}; z\right), \quad \Phi_n^\star(z) = \frac{(c)_n}{(b+1)_n}\, {}_2F_1\left(\begin{matrix}-n, 1+b\\ 1-n-c\end{matrix}; z\right) \tag{8.14}$$

These polynomials are biorthogonal on the unit circle

$$\int_{-\pi}^{\pi} \Phi_n(e^{i\theta})\Phi_m^\star(e^{-i\theta})e^{-i\theta(c-b)/2}|\sin(\theta/2)|^{c+b}d\theta = 0, \quad n \neq m \tag{8.15}$$

It is easy to see that the recurrent coefficients (8.12) can be obtained from (8.13) by the trivial rescaling in the limit $b \to \infty$. Omitting the simple technical details, we present here only the result. The polynomials corresponding to solution (8.11) or (8.12) are

$$\Phi_n(z;t) = \frac{t^n}{(c+1)_n}\, {}_2F_0(-n, 1+c; -z/t), \quad \Phi_n^\star(z;t) = t^{-n}(c)_n\, {}_1F_1\left(\begin{matrix}-n\\ 1-n-c\end{matrix}; tz\right) \tag{8.16}$$

These polynomials are biorthogonal on the unit circle

$$\int_{-\pi}^{\pi} \Phi_n(e^{i\theta})\Phi_m^\star(e^{-i\theta})e^{ic\theta}\exp(-te^{-i\theta}) = 0, \quad n \neq m. \tag{8.17}$$

Now let us consider the first Darboux-Bäcklund transformation (7.47)-(7.48) (see the previous section). We define $w(z)$ to be a weight function for the biorthogonal polynomials

$$\int \Phi_n(z)\Phi_m^\star(z^{-1})w(z)dz = 0, \quad n \neq m \tag{8.18}$$

Then, it is transformed under the first Darboux-Bäcklund transformation as

$$w(z;\tau+1) = \kappa z w(z;\tau) \tag{8.19}$$

where κ is a normalization constant.

In particular, in the case of the separate variables solution (8.11), this transformation is equivalent to shifting the parameter c: $\tilde{c} = c+1$, and, thus, one has a whole family of solutions depending on the both continuous and discrete times

$$b_n(t;\tau) = -\frac{tn}{(n+c_0+\tau)(n+c_0+\tau+1)}, \quad a_n(t;\tau) = -\frac{t}{n+c_0+\tau+1} \tag{8.20}$$

The property (8.19) is obviously fulfilled, which can be seen from orthogonality relation (8.17).

It is interesting to note that the Askey polynomials (8.14) themselves obey the discrete time Toda dynamics determined by the Darboux-Bäklund transformation provided that $c(\tau) = c_0 + \tau, \quad b(\tau) = b_0 - \tau$.

The general Darboux-Bäklund transformations of the biorthogonal polynomials will be considered in a separate publication.

9 Concluding remarks

In the present paper we considered some different representations of the relativistic Toda hierarchy, which are naively non-related to each other. However, from the point of view of studying the RTC hierarchy itself, the most promising representation is that describing the relativistic Toda hierarchy as a particular reduction of the two-dimensional Toda lattice hierarchy. However, even this quite large enveloping hierarchy is still insufficient.

Indeed, the RTC-reduction of the 2DTL being transparent on the level of the Lax operators can not be explicitly written in terms of the Clifford element. In other words, no reasonable description of the point of the Grassmannian is possible in the context of the usual (one-component) Toda lattice. Loosely speaking, the Toda lattice is too "rigid" to reproduce both the continuous and discrete flows of the RTC. Therefore, one can try to embed the RTC in a more general system which admits more natural reductions.

This is done in the forthcoming publication [19], where we show that the RTC has a nice interpretation if considering it as a simple reduction of the two-component KP (Toda) hierarchy. In this case, any dynamical variables acquire two discrete indices (we have two fermionic vacua in this picture) so that one can treat the second vacuum number as a discrete time which generates the sequence of the degenerate Bäcklund transformations (or, equivalently, the discrete evolution of the modified Toda-type). On the other hand, we need yet another discrete index in order to generate the whole discrete-time RTC. It turns out that the natural interpretation of this additional discretization is possible if one introduces the so-called Miwa variables (with the corresponding multiplicities) [6] after imposing the reduction conditions. The whole evolution of the RTC can be interpreted now as follows: the evolution along the additional vacuum generates the modified Toda equations while the evolution with respect to the multiplicities of the Miwa variables leads to general Darboux-Bäcklund transformations. This is the reason why the (properly) reduced two-component KP (Toda) is the true framework for description of RTC. Moreover, it turns out that, in the framework of the 2-component hierarchy, the continuous AKNS system, Toda chain hierarchy *and* the discrete AKNS (which is equivalent to the RTC how we proved in this paper) can be treated on equal footing. In particular, this means that all these systems corresponds to the same subspace in the Grassmannian.

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Appendix A. Evolution equations from the orthogonality conditions

Here we shall give some examples of deriving the evolution equations in the polynomial case starting from the orthogonality condition (3.1), (3.2). In order to keep the close connection with the general theory of the monic orthogonal polynomials, we write down here the spectral problem directly in terms of Φ_n and $\Phi_n^\star$:

$$z\Phi_n(z) = \Phi_{n+1}(z) - S_n h_n \sum_{k=0}^{n} \frac{S_{k-1}^\star}{h_k}\Phi_k(z) \equiv \mathcal{L}_{nk}\Phi_k(z) \tag{A.1}$$

$$z^{-1}\Phi_n^\star(z^{-1}) = \Phi_{n+1}^\star(z^{-1}) - S_n^\star h_n \sum_{k=0}^{n} \frac{S_{k-1}}{h_k}\Phi_k^\star(z^{-1}) \equiv h_n\overline{\mathcal{L}}_{kn}\frac{1}{h_k}\Phi_k^\star(z^{-1}) \tag{A.2}$$

Now the differentiation of (3.2) with respect to t_{-1} gives (for $n > k$) :

$$\left\langle \frac{\partial\Phi_n}{\partial t_{-1}}, \Phi_k^\star \right\rangle = < \Phi_n, z^{-1}\Phi_k^\star > \tag{A.3}$$

Using (A.2) it is easy to see that

$$\frac{\partial\Phi_n}{\partial t_{-1}} = \frac{h_n}{h_{n-1}}\Phi_{n-1} \equiv (1 - S_{n-1}S_{n-1}^\star)\Phi_{n-1} \equiv \frac{a_n}{b_n}\Phi_{n-1} \tag{A.4}$$

Similarly, the differentiation of (3.2) with respect to t_1 gives (for $n > k$) due to (A.1)

$$\left\langle \frac{\partial \Phi_n}{\partial t_1}, \Phi_k^\star \right\rangle = - < z\Phi_n, \Phi_k^\star >= S_n S_{k-1}^\star h_n \tag{A.5}$$

Therefore,

$$\begin{aligned} \frac{\partial \Phi_n}{\partial t_1} &= S_n h_n \sum_{k=0}^{n-1} \frac{S_{k-1}^\star}{h_k} \Phi_k(z) = \\ &= \frac{S_n}{S_{n-1}} \frac{h_n}{h_{n-1}} (\Phi_n - z\Phi_{n-1}) \equiv \; - b_n(\Phi_n - z\Phi_{n-1}) \end{aligned} \tag{A.6}$$

The equations (A.4), (A.6) are nothing but the simplest evolution equations (2.2), (2.16) which determine the RTC. Note that all the general equations (4.28) can be obtained in the same way. For example, the differentiation of (3.2) for $n > k$ with respect to t_m gives:

$$\left\langle \frac{\partial \Phi_n}{\partial t_m}, \Phi_k^\star \right\rangle = - < (\mathcal{L}^m)_{ns} \Phi_s, \Phi_k^\star >= -(\mathcal{L}^m)_{nk} h_k \theta(n-k-1) \tag{A.7}$$

and, therefore,

$$\frac{\partial \Phi_n}{\partial t_m} = -[(\mathcal{L}^m)_-]_{nk} \Phi_k \tag{A.8}$$

Similarly, the differentiation of (3.2) for $k = n$ yields

$$\frac{\partial h_n}{\partial t_m} \; = < z^m \Phi_n, \Phi_n^\star > \equiv < (\mathcal{L}^m)_{nk} \Phi_k, \Phi_n^\star > = \; (\mathcal{L}^m)_{nn} h_n \tag{A.9}$$

In the particular case of $m = 1$, one gets just $\mathcal{L}_{nn} = -S_n S_{n-1}^\star$, which leads to (3.23) with $\gamma = 0$.

Appendix B. Lax operators in 2DTL framework

In this appendix we are going to represent the spectral problem (3.7) arising from the orthogonal polynomials as the "usual" spectral problem. First of all, in the fast-decreasing case

$$S_n \to 0 \ , \qquad S_n^\star \to 0 \quad ; \quad n \to \pm\infty \tag{B.1}$$

one should prove that the solution to (3.7) with asymptotics (4.19) satisfies the spectral problem [17]

$$\mathcal{L}_{nk}\mathcal{P}_k(z) = z\mathcal{P}_n(z) \tag{B.2}$$

where $\mathcal{L}_{nk}$ is given by (4.17). Indeed, the equation (3.7) can be re-written in the form

$$\frac{1}{S_k h_k}\Big(\mathcal{P}_{k+1}(z) - z\mathcal{P}_k(z)\Big) - \frac{1}{S_{k-1}h_{k-1}}\Big(\mathcal{P}_k(z) - z\mathcal{P}_{k-1}(z)\Big) = \frac{S^\star_{k-1}}{h_k}\mathcal{P}_k(z) \tag{B.3}$$

In order to get (B.2), one should sum (B.3) from $k=-\infty$ to $k=n$ provided the following boundary condition is fulfilled (using that $h_k \to Const$ when $k\to-\infty$) :

$$\zeta_k(z) \equiv \frac{1}{S_k}\Big(\mathcal{P}_{k+1}(z) - z\mathcal{P}_k(z)\Big) \to 0 \ ; \qquad k\to-\infty \tag{B.4}$$

With the help of (4.21), (4.9), it is easy to see that coefficients $w_i(k)$ satisfy the recurrent relations

$$\frac{1}{S_k}\Big\{w_{i+1}(k+1) - w_{i+1}(k)\Big\} - \frac{1}{S_{k-1}}\Big\{w_i(k) - w_i(k-1)\Big\} = S^\star_{k-1}w_i(k-1) \quad ; \qquad w_0(k)\equiv 1 \tag{B.5}$$

i.e

$$\frac{1}{S_k}\Big\{w_i(k+1) - w_i(k)\Big\} = \sum_{j=1}^{i} S^\star_{k-j}w_{i-j}(k-j) \tag{B.6}$$

Thus, for any *fixed* i , the r.h.s of (B.6) tends to zero provided $S^\star_k w_i(k)\to 0$ as $k\to-\infty$. For sufficiently fast-decreasing S_k, $S^\star_k$, it is really true. Therefore, we see that all the coefficients of $\zeta_k(z)$ in (B.4) vanish in the limit $k\to-\infty$. Since solution $\mathcal{P}^{(1)}_n(z)$ is defined in the range of *large* values of the spectral parameter z , the whole function $\zeta_k(z)$ is convergent in the limit of large negative k and, thus, satisfies (B.4). Note that the same arguments fail in the case of solution (4.20) since it is defined at *small* values of the spectral parameter and the corresponding function $\bar\zeta_k(z)$ can not be properly defined as $k\to-\infty$ and, therefore, $\mathcal{P}^{(2)}_n(z)$ does not satisfy (B.2) in general.

To understand this situation in more details, it is instructive to consider the case of the forced hierarchy when

$$\begin{aligned} &S_k = S^\star_k \equiv 1 \ ; \quad k<0 \\ &S_k \ , \ S^\star_k \to 0 \ ; \quad k\to+\infty \end{aligned} \tag{B.7}$$

The first condition means also (see (3.6)) that

$$\frac{h_{k+1}}{h_k} = 0 \quad k<0 \quad h_0 \neq 0 \tag{B.8}$$

In the forced case, $\mathcal{P}^{(1)}_n(z)$ can be easily described. Indeed, it is a polynomial solution to (3.7) of the type (3.4) at $n>0$, $\mathcal{P}_0(z)=1$ and

$$\mathcal{P}^{(1)}_n(z) = (z+1)^n \ , \qquad n<0 \tag{B.9}$$

and it is well defined in the whole region of the spectral parameter. This solution is non-singular in the sense that

$$\frac{1}{h_n}\mathcal{P}^{(1)}_n(z) = 0 \ , \qquad n<0 \tag{B.10}$$

Thus, from (B.3) in the forced case, one can get the same spectral equation as in the fast-decreasing case, i.e

$$z\mathcal{P}_n(z) = \mathcal{P}_{n+1}(z) - S_n h_n \sum_{k=-\infty}^{n}\frac{S^\star_{k-1}}{h_k}\mathcal{P}_k(z) \equiv \mathcal{L}_{nk}\mathcal{P}_k(z) \tag{B.11}$$

where the summation actually goes from 0 to n due to (B.10).

On the other hand, the second independent solution to (3.7), $\mathcal{P}^{(2)}_n(z)$, is non-polynomial even in the forced case and can be treated as a formal series at small values of z only. Due to asymptotics (4.20), it is singular, i.e.

$$\frac{1}{h_n}\mathcal{P}^{(2)}_n(z) \neq 0 \ , \qquad n<0 \tag{B.12}$$

[17] See identification (3.56)

Moreover, putting

$$\mathcal{P}_n^{(2)}(z) = \frac{1}{h_n} x_n(z) \tag{B.13}$$

one can find out the exact answer

$$x_n(z) = z^n (1+z)^{-n-1} \quad , \qquad n < 0 \tag{B.14}$$

Now it is easy to see the reason why $\mathcal{P}_n^{(2)}(z)$ does not satisfy (B.11) in the generic situation. This is since the series

$$\sum_{k=-\infty}^{-1} \frac{S^\star_{k-1}}{h_k} \mathcal{P}_k^{(2)}(z) = \frac{1}{z} \sum_{k=0}^{\infty} \left(1 + \frac{1}{z}\right)^k \tag{B.15}$$

is divergent at small z though *formally* it is equal to -1 and this formal answer miraculously coincides with the *exact* expression which can be obtained for $\mathcal{P}_n^{(2)}(z)$ from (B.3) after summation from $k = 0$ to $k = n$ and using (B.13), (B.14). This is a typical example of how the formal analysis leads to a wrong conclusion and the analytical consideration is of great importance.

Instead of doing all this, one can prove in the same way that $\mathcal{P}_n^{(2)}(z) \equiv \overline{w}_n(z)$ (see (4.22), (4.10)) satisfies

$$\overline{\mathcal{L}}_{nk} \mathcal{P}_k(z) = z^{-1} \mathcal{P}_n(z) \tag{B.16}$$

where $\overline{\mathcal{L}}_{nk}$ is given by (4.18). It can be done by representing (3.7) in the form

$$\frac{1}{S_k}\Big(z^{-1}\mathcal{P}_{k+1}(z) - \mathcal{P}_k(z)\Big) - \frac{1}{S_{k-1}}\Big(z^{-1}\mathcal{P}_k(z) - \mathcal{P}_{k-1}(z)\Big) = S^\star_{k-1}\mathcal{P}_{k-1}(z) \tag{B.17}$$

and performing the summation from $k = n$ to $k = +\infty$. The only thing we need to prove is that $\mathcal{P}_k^{(2)}(z)$ satisfies the boundary condition

$$\bar{\zeta}_k(z) \equiv \frac{1}{S_k}\Big(z^{-1}\mathcal{P}_{k+1}(z) - \mathcal{P}_k(z)\Big) \to 0 \ ; \quad k \to +\infty \tag{B.18}$$

in the region of small z . This is in complete analogy with the proof of (B.4).

The rest of equations in (4.13), (4.15) can be derived from (4.23) using the same methods.

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